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Statistics of Persistent Events in the Binomial Random Walk: Will the Drunken Sailor Hit the Sober Man?

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The statistics of persistent events, recently introduced in the context of phase ordering dynamics, is investigated in the case of the one-dimensional lattice random walk in discrete time. We determine the survival probability of the random walker in the presence of an obstacle moving ballistically with velocity v, i.e., the probability that the random walker remains up to time n on the left of the obstacle. Three regimes are to be considered for the long-time behavior of this probability, according to the sign of the difference between v and the drift velocity $\bar{V}$ of the random walker. In one of these regimes ($v > \bar{V}$), the survival probability has a nontrivial limit at long times which is discontinuous at all rational values of v. An algebraic approach allows us to compute these discontinuities as well as several related quantities. The mathematical structure underlying the solvability of this model combines elementary number theory, algebraic functions, and algebraic curves defined over the rationals.

KEY WORDS: Random walk; large deviations; persistence; algebraic functions; periodic critical amplitudes.

1. INTRODUCTION

Consider an asymmetric binomial random walk on a one-dimensional lattice. In units of the lattice spacing, the steps ε_m performed by the walker at integer times $t=m$ are independent identically distributed random variables, with the binary law

$$\varepsilon_m = \begin{cases} -1 & \text{with probability} \quad p \\ +1 & \text{with probability} \quad q = 1-p \end{cases} \tag{1.1}$$

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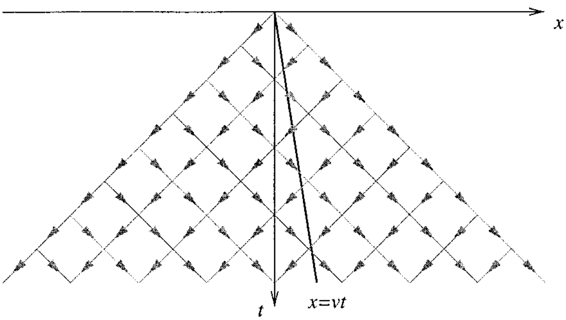

Fig. 1. Configuration space of the problem.

This paper is devoted to the analysis of the probability $F(n, v)$ that the walker remains, up to time n, on the left of an obstacle moving ballistically with velocity v. In a pictorial language, $F(n, v)$ is the probability that a sober man walking at constant speed v, and a drunken sailor stepping forward and backward erratically, both leaving a pub at some initial time, do not meet up to time n. Alternatively, $F(n, v)$ can be viewed as the survival probability of the walker in the presence of the obstacle. Surprisingly, as we shall see, this probability is highly nontrivial, especially as far as its v-dependence is concerned.

The quantity $F(n, v)$ also represents the probability that the path of the random walker in the (t, x)-plane remains, up to the integer time $t = n$, on the left of the straight wall $x = vt$. Figures 1 to 3 provide illustrations of these definitions.

This investigation was motivated by recent work on the statistics of persistent events in nonequilibrium statistical-mechanical systems undergoing phase ordering.[1] Persistent events are defined by a constraint on the past history of the system, to be satisfied up to time t, i.e., for all previous

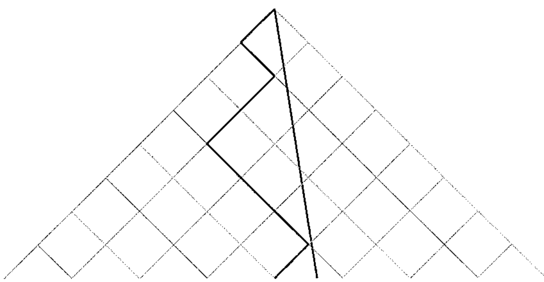

Fig. 2. A walk remaining on the left of the wall, thus contributing to the survival probability.

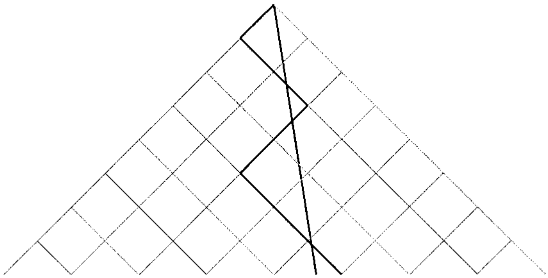

Fig. 3. A walk crossing the wall, thus not contributing to the survival probability.

times $0 \leqslant t' \leqslant t$. This concept can be simply illustrated on the example of a chain of Ising spins, starting from a random initial condition, and evolving under Glauber or heat-bath dynamics at zero temperature.[2] The persistence probability $R(t)$ is defined as the fraction of space which remained in the same phase up to time t, i.e., in the present case as the fraction of spins which did not flip up to time t.[3] For long times the persistence probability decays as

$$R(t) \sim t^{-\theta} \tag{1.2}$$

where θ is the persistence exponent.

The persistence exponent is actually only one member of a continuous family of exponents defined as follows.[1] Denoting by $\sigma(t)$ the spin at a given site, consider the local mean magnetization

$$M(t) = \frac{1}{t} \int_0^t \sigma(t')\, \mathrm{d}t' \tag{1.3}$$

This quantity is simply related to the fraction of time spent by the spin in the positive direction. Consider now the probability of *persistent large deviations* $R(t, x)$ above the level x (with $-1 \leqslant x \leqslant 1$), defined as the probability that $M(t)$ remained greater than x, for all times $0 \leqslant t' \leqslant t$. For the Glauber–Ising chain at zero temperature, this quantity is observed to decay algebraically at large times, with an exponent $\theta(x)$ continuously varying with x.[1] When $x = 1$, the usual persistence probability is recovered, so that $\theta(1) = \theta$. The exponent $\theta(x)$ appears as a first-passage exponent associated to persistent large deviations of $M(t)$.[1, 4, 5]

The interpretation of the stochastic process $\sigma(t)$ as the steps of a fictitious random walker naturally led us to the present study, where, for simplicity, we consider the concepts introduced above in the simplest case of a binomial random walk. In the present work, the analogue of $M(t)$ is the mean velocity V_n of the walker at time n,

$$V_n = \frac{x_n}{n} \tag{1.4}$$

where

$$x_n = \sum_{m=1}^{n} \varepsilon_m \tag{1.5}$$

is the position of the walker at time n, while $F(n, v)$ plays the role of the probability of persistent large deviations $R(t, x)$. Note a slight difference in the above definitions. In this work, $F(n, v)$ is the probability that the mean speed was always *less* than v, while in refs. 1, 4, and 5, $R(t, x)$ is the probability that the mean magnetization was always above x. This difference is harmless, as it just amounts to interchanging the probabilities p and q and changing the slope v into its opposite.

More precise definitions of the quantities considered in this work are as follows. First, the "one-time" distribution functions $P^{\pm}(n, v)$ of the mean velocity V_n at time n are defined for a given slope v as

$$\begin{aligned} P^{-}(n, v) &= \mathrm{Prob}\{V_n < v\} = \mathrm{Prob}\{x_n < nv\} \\ P^{+}(n, v) &= \mathrm{Prob}\{V_n \leqslant v\} = \mathrm{Prob}\{x_n \leqslant nv\} \end{aligned} \tag{1.6}$$

Then the survival probabilities $F^{\pm}(n, v)$ *up to* time n are defined as

$$\begin{aligned} F^{-}(n, v) &= \mathrm{Prob}\{V_m < v \text{ for } m = 1,..., n\} \\ &= \mathrm{Prob}\{x_m < mv \text{ for } m = 1,..., n\} \\ F^{+}(n, v) &= \mathrm{Prob}\{V_m \leqslant v \text{ for } m = 1,..., n\} \\ &= \mathrm{Prob}\{x_m \leqslant mv \text{ for } m = 1,..., n\} \end{aligned} \tag{1.7}$$

Two definitions are needed for each quantity, since V_n is a discrete random variable, so that $P^{\pm}(n, v)$ and $F^{\pm}(n, v)$ are in general discontinuous at any rational value of the slope v. Throughout this paper, quantities with the $+$ and the $-$ superscript are distinct from each other only when the slope v is rational. Equations involving $P^{\pm}(n, v)$, or $P(n, v)$ for short, are meant to hold separately for $P^{+}(n, v)$ and $P^{-}(n, v)$.

We find three regimes for the long-time behavior of $P^{\pm}(n, v)$ and $F^{\pm}(n, v)$, according to the sign of the relative velocity of the obstacle with respect to the drift velocity of the random walker

$$\bar{V} = \lim_{n \to \infty} V_n = \langle \varepsilon \rangle = 1 - 2p \tag{1.8}$$

with the following results.

• Large-deviation regime ($v < \bar{V}$): Both $P^{\pm}(n, v)$ and $F^{\pm}(n, v)$ decay exponentially in time, with a common entropy function $S(v)$, but different power-law prefactors, namely

$$P^{\pm}(n, v) \approx a_n^{\pm}(v)\, n^{-1/2} \mathrm{e}^{-nS(v)} \tag{1.9}$$

$$F^{\pm}(n, v) \approx b_n^{\pm}(v)\, n^{-3/2} \mathrm{e}^{-nS(v)} \tag{1.10}$$

• Marginal regime ($v = \bar{V}$): We have

$$P^{\pm}(n, v) \to \frac{1}{2} \tag{1.11}$$

$$F^{\pm}(n, v) \approx \frac{C^{\pm}(v)}{(\pi n)^{1/2}} \tag{1.12}$$

The limit value 1/2 of $P^{\pm}(n, v)$ is a consequence of the central limit theorem. The inverse-square-root decay of $F^{\pm}(n, v)$ is a known property of one-dimensional random walks.[6, 1, 4]

• Convergent regime ($v > \bar{V}$): $P^{\pm}(n, v)$ converge exponentially to one, while $F^{\pm}(n, v)$ admit nontrivial limits:

$$1 - P^{\pm}(n, v) \approx a_n^{\pm}(v)\, n^{-1/2} \mathrm{e}^{-nS(v)} \tag{1.13}$$

$$F^{\pm}(n, v) \to F^{\pm}(v) \tag{1.14}$$

These results were partially announced in ref. 1. The asymptotic behavior of the one-time distribution functions $P^{\pm}(n, v)$ are well-known properties of random walks,[6] while the long-time behavior of the survival probabilities $F^{\pm}(n, v)$ in these three regimes is the subject of the present study. To obtain this long-time behavior, we shall proceed in two steps.

A first level of description is provided in Section 2. We use a combinatorial result, originally due to Sparre Andersen, and proved in Appendix A, in order to derive the form of the estimates (1.10), (1.12), and (1.14). The prefactors $b_n^{\pm}(v)$ and $C^{\pm}(v)$, as well as the limit survival probabilities $F^{\pm}(v)$, are respectively given by Eqs. (2.22), (2.28), and (2.33). Such expressions hold for random walks with any distribution of steps ε_m, whose first two moments are finite. However they are only formal, in the sense that they do not lead to closed-form results in general.

A second, more refined level of description is provided by the rest of the paper, devoted to a detailed study of the quantities $b_n^{\pm}(v)$, $C^{\pm}(v)$, and $F^{\pm}(v)$. This investigation will allow us to unravel an unexpected richness in the statistics of persistent events in the present situation of the binomial random walk on the lattice of integers. More precisely, Section 3 contains basic concepts and results, emphasizing the central importance of sequences of positive integers $A_k^{\pm}$. Section 4 is devoted to the exposition of several methods (continuity of the path, probability flow, duality symmetry), which allow the determination of the integers $A_k^{\pm}$ and related quantities, in the case of an arbitrary slope v. The case of a rational slope v is dealt with in Section 5, where algebraic functions play a central role. Section 6 contains an investigation of the critical behavior of the survival probabilities, obtaining thus predictions in the three regimes. Further results on several specific situations, including the slopes of the form $v = 1 - 2/N$ or $v = -1 + 2/N$, are derived in Section 7, while Section 8 contains a discussion.

2. SPARRE ANDERSEN FORMALISM

As announced in the Introduction, this section is devoted to a first level of description of the one-time distribution functions $P^{\pm}(n, v)$ and of the survival probabilities $F^{\pm}(n, v)$.

We shall make an extensive use of a remarkable combinatorial result, originally due to Sparre Andersen.[7] The possibility of recasting the computation of $F^{\pm}(n, v)$ as a Sparre Andersen problem was uncovered in ref. 4. The result of Sparre Andersen is best expressed as an identity between the generating series of the $P^{\pm}(n, v)$ and of the $F^{\pm}(n, v)$, namely

$$f^{\pm}(z, v) = \sum_{n \geqslant 0} F^{\pm}(n, v)\, z^n = \exp\left(\sum_{n \geqslant 1} \frac{P^{\pm}(n, v)}{n}\, z^n\right) \tag{2.1}$$

This result can be found e.g. in the book by Feller.[6] Since Eq. (2.1) lacks a clear intuitive meaning, we give an elementary and self-contained combinatorial proof of it in Appendix A.

Taking the logarithmic derivative of both sides of Eq. (2.1), we obtain linear recursion relations for $F^{\pm}(n, v)$:

$$nF^{\pm}(n, v) = \sum_{m=0}^{n-1} P^{\pm}(n-m, v)\, F^{\pm}(m, v) \tag{2.2}$$

with $F^{\pm}(0, v) = 1$, hence

$$\begin{aligned}
F^{\pm}(1, v) &= P^{\pm}(1, v) \\
F^{\pm}(2, v) &= \tfrac{1}{2}((P^{\pm}(1, v))^2 + P^{\pm}(2, v)) \\
F^{\pm}(m, v) &= \tfrac{1}{2}((P^{\pm}(1, v))^3 + 3P^{\pm}(1, v)\, P^{\pm}(2, v) + 2P^{\pm}(3, v))
\end{aligned} \tag{2.3}$$

and so on.

In the present situation, the one-time distribution functions $P^{\pm}(n, v)$ can be evaluated in closed form as follows. We introduce the notations

$$p_c = \frac{1-v}{2}, \qquad q_c = 1 - p_c = \frac{1+v}{2} \tag{2.4}$$

Let k (respectively, $\tilde{k}$) be the number of steps to the left (respectively, to the right), i.e., the number of ε_m equal to -1 (respectively, to $+1$), among the first n steps. The probability distribution of the integer $k = 0,..., n$ reads

$$p_{n,k} = \binom{n}{k} p^k q^{n-k} \tag{2.5}$$

The position x_n of the particle at time n is related to k and $\tilde{k}$ by

$$n = k + \tilde{k}, \qquad x_n = \tilde{k} - k = n - 2k, \qquad k = \frac{n - x_n}{2}, \qquad \tilde{k} = \frac{n + x_n}{2} \tag{2.6}$$

The condition $V_n < v$ (respectively, $V_n \leqslant v$) is equivalent to $k > np_c$ (respectively, $k \geqslant np_c$), hence the result

$$P^{\pm}(n, v) = \sum_{k = \mathrm{Int}^{\pm}(np_c)+1}^{n} p_{n,k} \tag{2.7}$$

where we have defined the following two integer-part and fractional-part functions:

$$x = \mathrm{Int}^{\pm}(x) + \mathrm{Frac}^{\pm}(x), \qquad \text{with } \mathrm{Int}^{\pm}(x) \text{ integer and } \begin{cases} 0 \leqslant \mathrm{Frac}^{-}(x) < 1 \\ 0 < \mathrm{Frac}^{+}(x) \leqslant 1 \end{cases} \tag{2.8}$$

The random walk possesses a duality symmetry, defined by interchanging the probabilities p and q, and simultaneously changing the slope

v into its opposite $-v$, i.e., interchanging p_c and q_c. This amounts to considering the survival probability in the complementary domain, i.e., on the right of the wall shown in Fig. 1. Let us make the dependence of quantities on p explicit for a while, indicating the value of p after a semi-colon. Our conventions imply that the one-time distribution functions pertaining to the left of the wall and to the right of the wall read respectively $P^{\pm}(n, v; p)$ and $1 - P^{\mp}(n, -v; q)$. These quantities are clearly equal:

$$P^{\pm}(n, v; p) + P^{\mp}(n, -v; q) = 1 \tag{2.9}$$

As a consequence, the generating series $f^{\pm}(z, v)$ obeys the following identity:

$$f^{\pm}(z, v; p)\, f^{\mp}(z, -v; q) = \frac{1}{1-z} \tag{2.10}$$

More spectacular consequences of this duality symmetry will be investigated in Sections 4.3 and 5.3, by means of more powerful techniques.

We now turn to the investigation of the asymptotic behavior of the survival probabilities. As mentioned above, three regimes are to be considered, according to the sign of the relative velocity of the obstacle with respect to the drift velocity of the random walker [see Eq. (1.8)], namely

$$v - \bar{V} = v - (1 - 2p) = 2(p - p_c) \tag{2.11}$$

2.1. Large-Deviation Regime: $v < \bar{V}$ or $p < p_c$

This situation is typical of large deviations. The chance for the mean velocity of the random walker to deviate from its average is exponentially decreasing with time. More precisely, a careful treatment of the sums in Eq. (2.7), using Eq. (2.5) and Stirling's formula, leads to

$$P^{\pm}(n, v) \approx a_n^{\pm}(v)\, n^{-1/2} \mathrm{e}^{-nS(v)} \quad (v < \bar{V}) \tag{2.12}$$

[see Eq. (1.9)]. The associated entropy function (or large-deviation function) reads

$$\begin{aligned} S(v) &= p_c \ln \frac{p_c}{p} + q_c \ln \frac{q_c}{q} \\ &= \frac{1}{2}\left((1-v) \ln \frac{1-v}{2p} + (1+v) \ln \frac{1+v}{2q} \right) \end{aligned} \tag{2.13}$$

This function is positive, and it vanishes quadratically as $p \to p_c$, i.e., $\bar{V} \to v$, in agreement with the central limit theorem, according to

$$S(v) \approx \frac{(p-p_c)^2}{2p_c q_c} = \frac{(\bar{V}-v)^2}{2(1-v^2)} \tag{2.14}$$

In Eq. (2.12), the prefactors $a_n^{\pm}(v) = a^{\pm}(np_c)$ are v-dependent periodic functions of np_c, with unit period, which can be determined explicitly:

$$a^{\pm}(x) = \frac{(q_c p/(p_c q))^{1-\mathrm{Frac}^{\pm}(x)}}{(2\pi p_c q_c)^{1/2}\,(1 - q_c p/(p_c q))} \tag{2.15}$$

with the definition (2.8). The amplitude functions $a^{\pm}(x)$ can be alternatively expanded as Fourier series:

$$a^{\pm}(x) = \sum_{l=-\infty}^{\infty} \tilde{a}_l^{\pm}\, \mathrm{e}^{2\pi \mathrm{i} l x} \tag{2.16}$$

so that Eq. (2.12) can be recast as

$$P^{\pm}(n, v) \approx n^{-1/2} \sum_{l=-\infty}^{\infty} \tilde{a}_l^{\pm}\, \mathrm{e}^{(2\pi \mathrm{i} l p_c - S(v))\, n} \tag{2.17}$$

We remind for further reference the following general result, known as a Tauberian theorem.[6] If c_n are positive numbers, with asymptotic behavior

$$c_n \approx a n^{\gamma} (z_c)^{-n} \qquad (n \gg 1) \tag{2.18}$$

then the power series

$$f(z) = \sum_{n \geqslant 0} c_n z^n \tag{2.19}$$

defines an analytic function whose nearest singularity is at $z = z_c$, where it has a powerlaw singular part of the form

$$f_{\mathrm{sg}}(z) \approx a\, \Gamma(\gamma+1) \left(1 - \frac{z}{z_c}\right)^{-\gamma-1} \qquad (z \to z_c - 0) \tag{2.20}$$

Γ denoting Euler's gamma function. The reciprocal property, namely that Eq. (2.20) implies Eq. (2.18), holds if the sequence c_n is assumed to be smooth enough.

As a consequence of Eq. (2.20), the estimate (2.17) implies that the series $f^{\pm}(z, v)$ of Eq. (2.1) have square-root branch points at $z_l = \exp(S(v) - 2\pi i l p_c)$, where l runs over the integers. Using Eq. (2.18), we obtain the estimate

$$F^{\pm}(n, v) \approx b_n^{\pm}(v)\, n^{-3/2} e^{-nS(v)} \qquad (v < \bar{V}) \tag{2.21}$$

with $b_n^{\pm}(v) = b^{\pm}(np_c)$, where $b^{\pm}(x)$ are again periodic functions, with unit period, whose Fourier coefficients are related to those of $a^{\pm}(x)$ by

$$\tilde{b}_l^{\pm} = \tilde{a}_l^{\pm} f^{\pm}(z_l) \tag{2.22}$$

The survival probabilities $F^{\pm}(n, v)$ thus decay exponentially in the large-deviation regime, with the same entropy function as the one-time distribution functions $P^{\pm}(n, v)$, but with a different power of n [see Eqs. (1.9), (1.10)], multiplied by non-trivial periodic prefactors $b^{\pm}(np_c)$.

Figure 4 shows a plot of the periodic amplitudes $a(x)$ and $b(x)$, for $p = 0.3$, and for the "golden slope" v, such that p_c is the inverse golden mean, the most typical irrational:

$$p_c = \frac{\sqrt{5}-1}{2} = 0.618034, \qquad v = 2 - \sqrt{5} = -0.236068 \tag{2.23}$$

The data for $P(n, v)$ have been obtained by means of Eq. (2.7), and for $F(n, v)$ by solving the recursion relation (2.2). The amplitude $a(x)$ is found to be a pure exponential, in quantitative agreement with the prediction

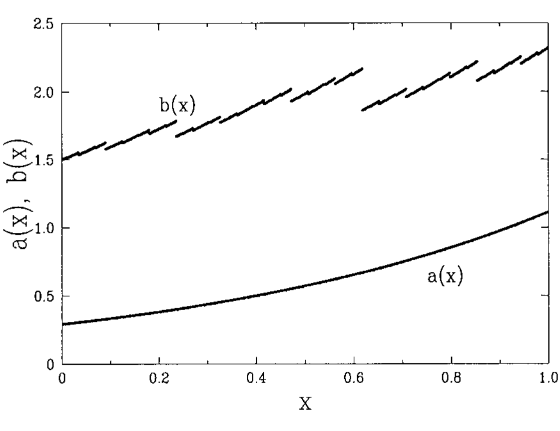

Fig. 4. Periodic amplitudes $a(x)$ and $b(x)$, for $p = 0.3$ and v equal to the golden slope.

(2.15). The amplitude $b(x)$ exhibits a rich structure, with discontinuities at all positive integer multiples of p_c modulo 1. The occurrence of these discontinuities can be explained as follows. Eq. (2.2) implies that

$$nF^{\pm}(n, v) = P^{\pm}(n, v) + F^{\pm}(1, v)\, P^{\pm}(n-1, v) + F^{\pm}(2, v)\, P^{\pm}(n-2, v) + \cdots \tag{2.24}$$

is a linear combination of $P^{\pm}(n, v)$, $P^{\pm}(n-1, v)$, $P^{\pm}(n-2, v)$, and so on, with n-independent coefficients. If we replace in Eq. (2.24) $P^{\pm}(n, v)$, $P^{\pm}(n-1, v)$, $P^{\pm}(n-2, v)$, by their asymptotic form (2.12), with $a_n^{\pm}(v) = a^{\pm}(np_c)$, we formally recover the result (2.21), with

$$\begin{aligned} b^{\pm}(x) = a^{\pm}(x) &+ F^{\pm}(1, v)\, \mathrm{e}^{S(v)} a^{\pm}(x - p_c) \\ &+ F^{\pm}(2, v)\, \mathrm{e}^{2S(v)} a^{\pm}(x - 2p_c) + \cdots \end{aligned} \tag{2.25}$$

Since the amplitude functions $a^{\pm}(x)$, given by Eq. (2.15), are discontinuous at $x = 0$ modulo 1, Eq. (2.25) shows that the amplitude functions $b^{\pm}(x)$ are discontinuous at all positive integer multiples of p_c modulo 1, confirming thus the observation made on Fig. 4.

2.2. Marginal Regime: $v = \bar{V}$ or $p = p_c$

In this regime, the drift velocity $\bar{V}$ of the random walker is equal to the velocity v of the obstacle. The one-time distribution functions $P^{\pm}(n, v)$ are expected to go to 1/2 as a consequence of the central limit theorem [see Eq. (1.11)].

A careful treatment of Eq. (2.7) leads to the more accurate estimate

$$P^{\pm}(n, v) \approx \frac{1}{2} + \left(\mathrm{Frac}^{\pm}(np_c) + \frac{p_c - 2}{3} \right) (2\pi n p_c q_c)^{-1/2} \qquad (n \gg 1) \tag{2.26}$$

The Sparre Andersen relation (2.1) then leads to the behavior[6]

$$f^{\pm}(z, v) \approx \frac{C^{\pm}(v)}{(1-z)^{1/2}} \qquad (z \to 1) \tag{2.27}$$

with

$$C^{\pm}(v) = \exp\left(\sum_{n \geqslant 1} \frac{P^{\pm}(n, v) - 1/2}{n} \right) \tag{2.28}$$

Equation (2.18) in turn leads to

$$F^{\pm}(n,v) \approx \frac{C^{\pm}(v)}{(\pi n)^{1/2}} \qquad (v=\bar{V}) \tag{2.29}$$

The survival probabilities $F^{\pm}(n,v)$ thus decay according to a universal inverse-square-root law in the marginal case, as already mentioned in Eq. (1.12). The associated amplitudes $C^{\pm}(v)$ are non-trivial functions of v, or equivalently of p_c, to which we shall come back in Section 6. The duality symmetry (2.10) implies the relation

$$C^{\pm}(v)\, C^{\mp}(-v)=1. \tag{2.30}$$

2.3. Convergent Regime: $v>\bar{V}$ or $p>p_c$

In this regime, the drift velocity $\bar{V}$ of the random walker belongs to the domain in which the mean velocities V_n are constrained. These constraints are therefore less and less stringent as time goes on. Indeed $P^{\pm}(n,v)$ go to unity exponentially fast, with $1-P^{\pm}(n,v)$ given by a large-deviation expression [see Eq. (1.13)].

The Sparre Andersen relation (2.1) now leads to the estimate

$$f^{\pm}(z,v) \approx \frac{F^{\pm}(v)}{1-z} \qquad (z\to 1) \tag{2.31}$$

In other words, the survival probabilities converge toward nontrivial limits for infinite times:

$$F^{\pm}(v)=\lim_{n\to\infty} F^{\pm}(n,v) \tag{2.32}$$

hence the word "convergent regime." The limit survival probabilities have the formal expressions

$$F^{\pm}(v)=\exp\left(-\sum_{n\geqslant 1}\frac{1-P^{\pm}(n,v)}{n}\right) \qquad (v>\bar{V}) \tag{2.33}$$

The limit probabilities $F^{\pm}(v)$ have in general a very rich dependence on v and p, which is not apparent in the formal expressions (2.33), and which will be investigated in the following sections. For the time being, we want to underline the existence of a universal relationship between the p-dependence of $F^{\pm}(v)$ as $p\to p_c+0$ and the amplitudes $C^{\pm}(v)$ corresponding to the marginal regime ($p=p_c$). For $p-p_c$ small and positive, the one-time distribution functions $P^{\pm}(n, v)$ can be approximated by their values at $p = p_c$ as long as $n(p - p_c) \ll 1$, while they can be estimated from the central limit theorem for $n(p - p_c) \sim 1$:

$$P^{\pm}(n, v) \approx \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{n(p - p_c)}{2^{3/2} p_c q_c}\right)\right] \tag{2.34}$$

where erf is the error function. By inserting the above estimate into Eq. (2.33), and splitting the sum over n into three sums, respectively corresponding to the ranges $1 \leqslant n \leqslant n_0$, $n_0 < n \leqslant n_1$, and $n > n_1$, with $1 \ll n_0 \ll 1/(p - p_c) \ll n_1$, we obtain after some manipulations

$$F^{\pm}(v) \approx C^{\pm}(v)\left(\frac{2}{p_c q_c}\right)^{1/2}(p - p_c) \approx C^{\pm}(v)\left(\frac{2}{1 - v^2}\right)^{1/2}(v - \bar{V}) \tag{2.35}$$

The survival probabilities $F^{\pm}(v)$ thus vanish linearly as $p \to p_c$, with amplitudes proportional to the constants $C^{\pm}(v)$, characteristic of the fall off (2.29) in the marginal regime ($p = p_c$).

The quantities introduced to describe the asymptotic long-time behavior of the survival probabilities $F^{\pm}(n, v)$ in the three different regimes, namely the periodic functions $b^{\pm}(x, v)$, the constants $C^{\pm}(v)$, and the limit survival probabilities $F^{\pm}(v)$, bear a highly nontrivial dependence on the parameters p and v, which is hidden in their formal expressions (2.22), (2.28), and (2.33). The rest of this paper is devoted to an investigation of all these quantities, including their dependence on p and v. Throughout the following, when the context permits, we shall suppress the explicit dependence on v, writing e.g. $F^{\pm}$ for $F^{\pm}(v)$.

3. BASIC CONCEPTS

3.1. The Survival Probability as a Distribution Function

Define the maximal velocity $V_{\max}$ of a given (infinite) random walk as the supremum of all the instantaneous mean velocities V_n:

$$V_{\max} = \sup_n V_n \tag{3.1}$$

This quantity is a random variable, as it depends on the walk under consideration. The survival probabilities $F^{\pm}(v)$ can be interpreted as the distribution functions of this random variable, for a fixed value of p. We have indeed

$$F^{-}(v) = \operatorname{Prob}\{V_{\max} < v\}, \qquad F^{+}(v) = \operatorname{Prob}\{V_{\max} \leqslant v\} \tag{3.2}$$

With probability one the random variable $V_{\max}$ is rational, and it lies in the range $\bar{V} < V_{\max} \leqslant 1$. These properties originate in the following two facts. The V_n converge with probability one to the drift velocity $\bar{V}$, and the differences $V_n - \bar{V}$ take both signs. As a consequence, with probability one there is some finite n such that $V_{\max} = V_n$. We have therefore

$$F^{\pm}(v) = 0 \quad (v \leqslant \bar{V}), \qquad F^{\pm}(v) = 1 \quad (v > 1) \tag{3.3}$$

The lower edge ($v = \bar{V}$) corresponds to the marginal case studied in Section 2.2, while the upper edge ($v = 1$) will be investigated in Section 7.1.

Whenever the slope v is irrational, we have $F^+(v) = F^-(v)$, and the distribution function is continuous. On the contrary, when v is rational, quantities with superscripts $+$ and $-$ are different from each other in general. The discontinuity $\Pi(v)$ of the distribution function is nothing but the probability that the maximal velocity $V_{\max}$ assumes the value v:

$$\Pi(v) = F^+(v) - F^-(v) = \mathrm{Prob}\{V_{\max} = v\} \tag{3.4}$$

and we have

$$F^+(v) = \lim_{w \to v+0} F(w), \qquad F^-(v) = \lim_{w \to v-0} F(w) \tag{3.5}$$

The above quantities can be related as follows. Let $Q(v)$ be the probability that the random walker makes its first step to the left and then touches the wall at least once:

$$Q(v) = \sum_{n \geqslant 2} \mathrm{Prob}\{V_m < v \text{ for } m = 1,..., n-1 \text{ and } V_n = v\} \tag{3.6}$$

This quantity is non-zero for any rational v in the range $-1 < v \leqslant 1$. We can split any walk contributing to $Q(v)$ into two independent walks: a finite walk from the origin to the first coincidence time n such that $V_n = v$, and the rest, which is an infinite walk. Consider a walk contributing to $F^+(v)$. It either contributes to $F^-(v)$ or to $\Pi(v)$. In the second case, we have $V_{\max} = v$, and with probability one there is some n such that $V_n = v$ for the first time. The complete walk splits into two independent walks: a finite walk from the origin to the first coincidence, and the rest, which is an infinite walk contributing to $F^+(v)$. So, in order to go from $Q(v)$ to the difference $F^+(v) - F^-(v)$, one just replaces the arbitrary infinite part by a walk contributing to $F^+(v)$. In other words, we have

$$\Pi(v) = F^+(v) - F^-(v) = Q(v)\, F^+(v), \tag{3.7}$$

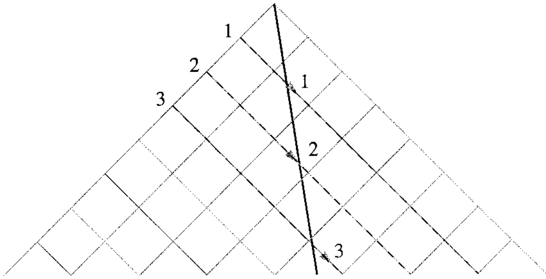

Fig. 5. Labeling of crossing edges by an integer $k=1, 2, 3$, equal to the number of steps of the walker to the left.

with this quantity vanishing for $v \leqslant \bar{V}$ and being positive for $v > \bar{V}$, or else

$$F^{-}(v) = (1 - Q(v))\, F^{+}(v) \tag{3.8}$$

The case of a rational slope v will be investigated in Section 4. It turns out that some general combinatorial statements can be conveniently presented by keeping v arbitrary. This is the purpose of the rest of Section 3.

3.2. Truncated Pascal Triangle

The oriented lattice edges crossing the wall from left to right will play an important role in what follows. Indeed the walks which contribute to the survival probability, i.e., those which remain on the left forever, are those which start to the left and never pass through a crossing edge. We assume for a while that v is irrational. As shown in Fig. 5, we index the crossing edges by the integer $k \geqslant 1$, i.e., by the number of steps to the left, defined in Eq. (2.6). The endpoint of the kth crossing edge has coordinates $(n, x) = (n_k, n_k - 2k)$, where n_k is defined by the inequalities $k/p_c < n_k < k/p_c + 1$, or equivalently

$$n_k = 1 + \mathrm{Int}(k/p_c) \tag{3.9}$$

When v is rational, we have to be more careful and define two sequences. The edges corresponding to $F^{-}(v)$ may end but not start on the wall, while those corresponding to $F^{+}(v)$ may start but not end on the wall, hence the prescriptions

$$n_k^{-} = 1 + \mathrm{Int}^{+}(k/p_c), \qquad n_k^{+} = 1 + \mathrm{Int}^{-}(k/p_c) \tag{3.10}$$

Hence, if v is rational, any construction involving the sequence of crossing edges will give two different outputs in general, depending on which family of crossing edges is considered. In this case, we may, and shall sometimes, start the sequence of crossing edges with $k=0$.

Let us emphasize that our problem can be recast as a purely combinatorial one. We assume again for simplicity that v is irrational. Let $\mathcal{N}_{n,k}$ be the number of n-step walks starting from the origin, remaining on the left of the wall, and ending at the point $(n, n-2k)$, with the notations (2.6). The numbers $\mathcal{N}_{n,k}$ obey a recursion relation very similar to that obeyed by the binomial coefficients $\binom{n}{k}$:

$$\mathcal{N}_{n,k} = \begin{cases} \mathcal{N}_{n-1,k-1} + \mathcal{N}_{n-1,k} & (k > np_c) \\ 0 & (k < np_c) \end{cases} \tag{3.11}$$

with the initial condition $\mathcal{N}_{0,0}=1$, hence in particular $\mathcal{N}_{n,n}=1$. The recursion (3.11) leads to a double array of integers, shown in Fig. 6, that we refer to as a truncated Pascal triangle.

If the slope v is rational, i.e., p_c is rational, one has to consider two series of integers, $\mathcal{N}^{\pm}_{n,k}$, which obey the recursion relations

$$\begin{aligned} \mathcal{N}^{+}_{n,k} &= \begin{cases} \mathcal{N}^{+}_{n-1,k-1} + \mathcal{N}^{+}_{n-1,k} & (k \geqslant np_c) \\ 0 & (k < np_c) \end{cases} \\ \mathcal{N}^{-}_{n,k} &= \begin{cases} \mathcal{N}^{-}_{n-1,k-1} + \mathcal{N}^{-}_{n-1,k} & (k > np_c) \\ 0 & (k \leqslant np_c) \end{cases} \end{aligned} \tag{3.12}$$

again with $\mathcal{N}^{\pm}_{0,0}=1$.

As the weight of a finite walk only depends on its endpoint, the integers $\mathcal{N}_{n,k}$ formally contain all the relevant information for the computation of

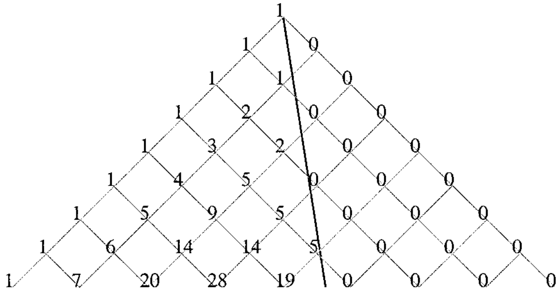

Fig. 6. Combinatorial approach: truncated Pascal triangle.

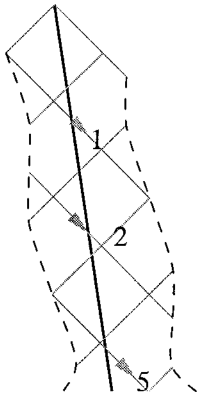

Fig. 7. The probability flow is characterized by integers $A_k^{\pm}$, living on the crossing edges [cf. Fig. 5]. In the example, we have $A_1^{\pm}=1$, $A_2^{\pm}=2$, $A_3^{\pm}=5$, and so on.

the survival probability. More powerful techniques allowing to extract this information will be exposed in the following.

3.3. Probability Flow Equation

The recursion relations (3.11), (3.12) for the integers $\mathcal{N}^{\pm}_{n,k}$ are different from those for the binomial coefficients at the crossing edges. This indicates that there is a probability flow through the wall. To analyze this probability flow, we define the numbers $A_k^{\pm}$ as the elements of the truncated Pascal triangle at the beginning of a crossing edge:

$$A_k^{\pm} = \mathcal{N}^{\pm}_{n_k - 1, k} \tag{3.13}$$

Equations (3.11), (3.12) imply that $A_k^{\pm}$ walks are absorbed at the kth crossing edge. In order to count them with the right probability [see Eqs. (3.14), (3.15) below], we have to put these numbers at the end of the crossing edges, obtaining thus the picture shown in Fig. 7. In the above example, we get the sequence 1, 2, 5, 19,...

This construction leads to the most important (though elementary) equation of this paper. The probability to begin with a left step is p, while $A_k^{\pm}$ walks are lost at the kth crossing edge, each with a probability $p^k q^{n_k^{\pm}-k}$, hence the *probability flow equation*

$$F^{\pm}(n, v) = p - \sum_{k=1}^{\mathrm{Int}^{\pm}(np_c)} A_k^{\pm} p^k q^{n_k^{\pm}-k} \tag{3.14}$$

where we have used the equivalence between the inequalities $n_k^{\pm} \leqslant n$ and $k \leqslant \mathrm{Int}^{\pm}(np_c)$. Taking the $n \to \infty$ limit, we obtain the following flow equation

$$F^{\pm} = p - \sum_{k \geqslant 1} A_k^{\pm} p^k q^{n_k^{\pm} - k} \tag{3.15}$$

for the limit survival probabilities $F^{\pm}$, which still depend on p and v.

The flow equations (3.14) and (3.15) will be used several times in the following. In particular, Eq. (3.15) will be used in Section 4 in order to derive the second and third recursion relations satisfied by the sequences $A_k^{\pm}$. Before we turn to these matters, we can obtain some information on the analytic structure of $F^{\pm}$ in p. Equation (3.15) shows that the expression

$$\sum_{k \geqslant 1} A_k^{\pm} p^k q^{n_k^{\pm} - k} \tag{3.16}$$

with $q = 1 - p$, is a convergent sum of positive numbers for any p in the interval $[0, 1]$. Using the definition (3.9), we have

$$p^k q^{n_k^{\pm} - k} = z(p)^k \, q^{1 - \mathrm{Frac}^{\mp}(k/p_c)} \tag{3.17}$$

with

$$z(p) = p q^{q_c/p_c} \tag{3.18}$$

As the second factor $q^{1 - \mathrm{Frac}^{\mp}(k/p_c)}$ is bounded, we can concentrate on the first factor $z(p)^k$. The maximum of the function $z(p)$ for p in $[0, 1]$ is reached for $p = p_c$. Hence the sum (3.16) is a holomorphic function of p in a neighborhood of the origin, and it can be expanded term by term as long as $|z(p)| < z(p_c)$. Suppose now $p > p_c$. Equation (3.17) implies $p^k q^{n_k^{\pm} - k} < p_c^k q_c^{n_k^{\pm} - k}$ for each k, so that

$$F^{\pm} > p - \sum_{k \geqslant 1} A_k^{\pm} p_c^k q_c^{n_k^{\pm} - k} = p - p_c \tag{3.19}$$

This property gives a proof of the fact, announced before, that $F^{\pm} > 0$ for $p > p_c$. This also shows that the sum (3.16) is not analytic at $p = p_c$, because if it were so the equality $F^{\pm} = 0$, which holds for $p \leqslant p_c$, could be continued across $p = p_c$.

4. ARBITRARY SLOPES: RECURSION RELATIONS

This section is devoted to the investigation of the sequences of integers $A_k^{\pm}$. The slope v is an arbitrary number throughout this section, so that we shall omit the superscripts $\pm$, for brevity.

4.1. Continuity of the Path

By expressing the continuity of the path of the random walker, we shall obtain the first recursion relation defining the integers A_k.

Consider a walk of length n_k that starts to the left and terminates at the end of the kth crossing edge. There are $\binom{n_k-1}{k-1}$ such walks. Either this walk crosses the wall for the first time along the kth crossing edge (there are A_k such walks), or it has crossed the wall for the first time along some earlier crossing edge, say the lth one. In the second case, the full walk is the concatenation of one of the A_l walks that cross for the first time along the lth crossing edge, and of any walk from (n_l, l) to (n_k, k) (there are $\binom{n_k-n_l}{k-l}$ such walks). We thus obtain our first recursion relation for the sequence of integers A_k:

$$\binom{n_k-1}{k-1}=\sum_{l=1}^{k}\binom{n_k-n_l}{k-l}A_l=A_k+\sum_{l=1}^{k-1}\binom{n_k-n_l}{k-l}A_l \tag{4.1}$$

Equation (4.1) clearly leads to integer values for A_k, but not obviously to positive ones. The positivity of the A_k can be checked on the first few of them, i.e.,

$$A_1=1, \qquad A_2=n_1-1, \qquad A_3=\tfrac{1}{2}(n_1-1)(2n_2-n_1-2), \qquad \text{etc.} \tag{4.2}$$

The recursion relation (4.1) looks like a convolution. It fails however to be an exact convolution, because $n_k-n_l\neq n_{k-l}$ in general. Nevertheless the difference between these integers is small (zero or one in absolute value).

4.2. Probability Flow

The second recursion relation for the integers A_k relies on the probability flow equation (3.15), and on the analyticity at small p of the sum (3.16), proved at the end of Section 3.3. We know that $F=0$ for fixed v and small enough p. Hence we can apply contour integrals to both sides of the relation

$$p=\sum_{k\geqslant 1}A_k p^k q^{n_k-k} \tag{4.3}$$

To be more precise, let $l\geqslant 1$ be an integer, and $c_l(p)$ be a holomorphic function in a neighborhood of the origin, with a Taylor expansion starting as $c_l(p) = p^{l+1} + \cdots$. Integrating along a small contour around the origin, we have

$$\oint \frac{\mathrm{d}p}{2\pi \mathrm{i} c_l(p)}\, p = \sum_{k \geqslant 1} A_k \oint \frac{\mathrm{d}p}{2\pi \mathrm{i} c_l(p)}\, p^k q^{n_k - k}$$
$$= A_l + \sum_{k=1}^{l-1} A_k \oint \frac{\mathrm{d}p}{2\pi \mathrm{i} c_l(p)}\, p^k q^{n_k - k} \tag{4.4}$$

Indeed the integral in the two rightmost sides equals 0 for $k > l$, and 1 for $k = l$.

We can take any sequence of functions $c_l(p)$, and get a corresponding recursion relation which determines the A_k. The following three cases will prove useful in the sequel.

- The most obvious candidate is

$$c_l(p) = p^{l+1} \tag{4.5}$$

which leads to our second recursive definition of the A_k:

$$\delta_{k,1} = A_k + \sum_{l \leqslant k-1,\, k \leqslant n_l} (-1)^{k-l} \binom{n_l - l}{k - l} A_l \tag{4.6}$$

where $\delta_{k,l}$ is the Kronecker symbol. Again, the integrality of the A_k is obvious, but their positivity is not.

- Another simple choice reads

$$c_l(p) = p^{l+1} q^{n_l - l + 1} \tag{4.7}$$

Surprisingly enough, we recover our first recursion formula (4.1), expressing the continuity of the path. This example illustrates the generality of Eq. (4.4).

- Let us come back to Eq. (3.17), which shows that the sum (3.16) is almost an entire series in $z(p)$, with the definition (3.18), up to a small positive power of q, with an exponent in the range $[0, 1]$. If we now divide expression (3.16) by q, we have again almost an entire series in $z(p)$, up to a small negative power of q, with an exponent in the range $[-1, 0]$. It is therefore natural to evaluate the contour integrals of Eq. (4.4) with the weights

$$\frac{\mathrm{d}p}{c_l(p)} = \frac{\mathrm{d}z}{z^{l+1}} \quad \text{and} \quad \frac{\mathrm{d}p}{c_l(p)} = \frac{\mathrm{d}z}{z^{l+1}(1 - p(z))} \tag{4.8}$$

where $p(z)$ is the inverse function of $z(p)$, defined in Eq. (3.18). After somewhat lengthy computations, which boil down to binomial expansions, we respectively obtain

$$\frac{\Gamma(k/p_c-1)}{k!\,\Gamma(kq_c/p_c)}=A_k-\sum_{l=1}^{k-1}\frac{\Gamma(k/p_c-n_l)}{(k-l)!\,\Gamma(kq_c/p_c+1-n_l+l)}\,(1-\mathrm{Frac}(l/p_c))\,A_l$$
$$\frac{\Gamma(k/p_c+1)}{k!\,\Gamma(kq_c/p_c+2)}=A_k+\sum_{l=1}^{k-1}\frac{\Gamma(k/p_c-n_l+1)}{(k-l)!\,\Gamma(kq_c/p_c+2-n_l+l)}\,\mathrm{Frac}(l/p_c)\,A_l \tag{4.9}$$

The recursion relations (4.9) seem rather complicated, so that even the integrality of the A_k is not obvious. Equations (4.9) are nevertheless of interest. Indeed all the coefficients in the sums have the same sign, since the A_k are positive. We thus obtain the following bounds

$$\frac{\Gamma(k/p_c-1)}{k!\,\Gamma(kq_c/p_c)}\leqslant A_k\leqslant\frac{\Gamma(k/p_c+1)}{k!\,\Gamma(kq_c/p_c+2)} \tag{4.10}$$

The case where $p_c=1/N$, with N an integer, is of special interest. In this situation, $l/p_c=lN$ is an integer for any l. Re-introducing the $\pm$ superscripts for a while, we thus have $\mathrm{Frac}^+(l/p_c)=1$ if we are interested in F^-, and $\mathrm{Frac}^-(l/p_c)=0$ if we are interested in F^+. As a consequence, the A_k^- saturate the lower bound of Eq. (4.10), while the A_k^+ saturate the upper bound. We shall recover these properties in Section 7.3 with more powerful techniques.

In the general case, we set

$$A_k^\pm=B_k^\pm\,\frac{\Gamma(k/p_c)}{k!\,\Gamma(kq_c/p_c+1)} \tag{4.11}$$

Equation (4.10) shows that the prefactors $B_k^\pm$ are bounded. In other words, using again Stirling's formula, we obtain the estimates

$$A_k^\pm\approx B_k^\pm\,\frac{p_c}{(2\pi q_c k^3)^{1/2}}\,(p_c q_c^{q_c/p_c})^{-k}\qquad(k\gg 1) \tag{4.12}$$

The prefactors $B_k^\pm$ obey the simple bounds

$$q_c\leqslant B_k^\pm\leqslant\frac{1}{q_c} \tag{4.13}$$

We shall come back in Section 6 to these prefactors, which are nontrivial in general.

4.3. Duality

The recursion relations derived in the previous section are based on the fact that $F=0$ for small enough p. We now aim at using the vanishing of F on the full interval $0 \leqslant p \leqslant p_c$. We call the following approach duality, as in the beginning of Section 2, since it relates the survival probability in presence of the obstacle with slope v to that with slope $-v$. In Section 5.3 we shall give a simple combinatorial argument explaining why the slopes v and $-v$, i.e., p_c and q_c are related, at least for rational v.

We start again from the identity (4.3),

$$p - \sum_{k \geqslant 1} A_k p^k q^{n_k - k} = 0 \qquad (0 \leqslant p \leqslant p_c) \tag{4.14}$$

and the similar equation for the slope $-v$, with integers $\tilde{A}_k$,

$$p - \sum_{k \geqslant 1} \tilde{A}_k p^k q^{\tilde{n}_k - k} = 0 \qquad (0 \leqslant p \leqslant p_c) \tag{4.15}$$

By substituting $q = 1 - p$ for p in this last equation, we get

$$q - \sum_{k \geqslant 1} \tilde{A}_k q^k p^{\tilde{n}_k - k} = 0 \qquad (p_c \leqslant p \leqslant 1) \tag{4.16}$$

Multiplying Eqs. (4.14), (4.16), we obtain

$$\left(p - \sum_{k \geqslant 1} A_k p^k q^{n_k - k}\right)\left(q - \sum_{k \geqslant 1} \tilde{A}_k q^k p^{\tilde{n}_k - k}\right) = 0 \qquad (0 \leqslant p \leqslant 1) \tag{4.17}$$

For the rest of this section, we assume that v is irrational, while the case where v is rational is treated in Section 5.3. For v irrational and $k \geqslant 1$, the integer n_k corresponding to the kth crossing with the wall at slope v is defined by the inequalities $0 < p_c n_k - k < p_c$, while the integer $\tilde{n}_k$ corresponding to the kth crossing with the wall at slope $-v$ is defined by the inequalities $-q_c < p_c \tilde{n}_k - (\tilde{n}_k - k) < 0$. Hence the fractional parts of $p_c n_k$ and $p_c \tilde{n}_k$ live in the disjoint intervals $[0, p_c]$ and $[-q_c, 0]$. Consequently, the set of the n_k and the set of the $\tilde{n}_k$ have no integer in common. Note that n_k and $\tilde{n}_k$ are both larger than 1.

Conversely, for $n > 1$, there is a unique integer k_n such that $-q_c < np_c - k_n < p_c$. We set $\theta_n = np_c - k_n$. Equivalently, with the definitions (2.8), we have

$$k_n = 1 + \mathrm{Int}((n-1)\, p_c), \qquad \theta_n = -q_c + \mathrm{Frac}((n-1)\, p_c) \tag{4.18}$$

Let I^+ be the set of integers $n > 1$ such that $\theta_n > 0$. For such an n, there is an integer $k \geqslant 1$ such that $n = n_k$, namely $k = k_n$, and we have

$$n \in I^+: \quad \theta_n = p_c \left(1 - \mathrm{Frac}\left(\frac{k_n}{p_c}\right)\right) \tag{4.19}$$

Similarly, let I^- be the set of integers $n > 1$ such that $\theta_n < 0$. For such an n, there is an integer $k \geqslant 1$ such that $n = \tilde{n}_k$, namely $k = n - k_n$ and we have

$$n \in I^-: \quad \theta_n = -q_c \left(1 - \mathrm{Frac}\left(\frac{n - k_n}{q_c}\right)\right) \tag{4.20}$$

The above two sets are a partition of the integers $n > 1$. We can therefore define a sequence of integers $\bar{A}_n$ for $n > 1$ by

$$\bar{A}_n = \begin{cases} A_{k_n} & \text{if} \quad n \in I^+ \\ \tilde{A}_{n-k_n} & \text{if} \quad n \in I^- \end{cases} \tag{4.21}$$

A last geometrical observation is in order. It follows from the definition of the sequence k_n that $k_n \leqslant k_{n+1} \leqslant k_n + 1$. Hence the sequence

$$X_n = n - 2k_n \tag{4.22}$$

defines a canonical walk, shown in Fig. 8.

This walk is closest to the wall in a well-defined sense: the point $(n, n - 2k_n)$ is the endpoint of an edge crossing the wall, either from left to right or from right to left. Equivalently, the next step of the canonical walk

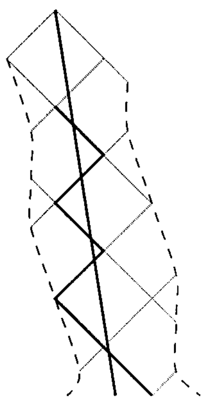

Fig. 8. The canonical walk.

is to the left if it starts to the right of the wall, and vice versa. We identify the symbol X_n with the polynomial

$$X_n = p^{k_n} q^{n-k_n} \tag{4.23}$$

whose total degree is n, and whose difference of partial degrees in p and q gives the position X_n, as in Eq. (4.22).

With the notations (4.21), (4.23), Eq. (4.17) can be recast as

$$\left(1 - q - \sum_{n \in I^+} \bar{A}_n X_n\right)\left(q - \sum_{n \in I^-} \bar{A}_n X_n\right) = 0 \qquad (0 \leqslant p \leqslant 1) \tag{4.24}$$

Let us expand the product. It is easy to check that if n and m have different I-signs, then $X_n X_m = X_{n+m}$. Moreover, if $n \in I^+$, $pX_n = X_{n+1}$, and if $n \in I^-$, $qX_n = X_{n+1}$. Finally, $pq = X_2$. We set $\bar{A}_1 = 1$, and $I_0^+ = I^+ \cup \{1\}$ and $I_0^- = I^- \cup \{1\}$. We obtain after some algebra

$$\sum_{n > 1} \bar{A}_n X_n = \sum_{m \in I_0^-, n \in I_0^+} \bar{A}_m \bar{A}_{n+m} \qquad (0 \leqslant p \leqslant 1) \tag{4.25}$$

A heuristic term-by-term identification leads to a third recursive definition of the integers $\bar{A}_n$:

$$\bar{A}_l = \sum_{\substack{m \in I_0^-, n \in I_0^+ \\ m+n=l}} \bar{A}_m \bar{A}_n \qquad (l \geqslant 2) \tag{4.26}$$

An algebraic proof of this relation will be given in Section 5.3.

5. RATIONAL SLOPES: ALGEBRAIC APPROACH

In this section we investigate in further detail the outcomes of the techniques exposed in Section 4, in the case where v is rational. Algebraic functions play a central role in this analysis.

We write a rational slope in terms of two relatively prime integers M, $\tilde{M} \geqslant 1$ as

$$N = M + \tilde{M}, \qquad v = \frac{\tilde{M} - M}{N}, \qquad p_c = \frac{M}{N}, \qquad q_c = \frac{\tilde{M}}{N} \tag{5.1}$$

The integers $n_k^{\pm}$, defined in Eq. (3.10), are skew periodic, in the sense that

$$n_{k+M}^{\pm} = n_k^{\pm} + N \tag{5.2}$$

and we have

$$n_k^- = n_k^+ \quad (k = 1, \ldots, M-1), \qquad n_M^- = N, \quad N_M^+ = N+1 \tag{5.3}$$

We recall that, for each rational slope v, we have to consider two different sequences of integers, $A_k^\pm$, associated with the survival probabilities $F^\pm$.

5.1. Continuity of the Path

As a consequence of the property (5.2) of the integers $n_k^\pm$, the recursion relation (4.1) can be split into M blocks, one for each value of k modulo M.

We introduce the following formal power series in t:

$$F_k^\pm(t) = \sum_{K \geqslant 0} A_{KM+k}^\pm t^K \qquad (k = 1, \ldots, M) \tag{5.4}$$

$$G_k^\pm(t) = \sum_{K \geqslant 0} \binom{KN + n_k^\pm - 1}{KM + k - 1} t^k \qquad (k = 1, \ldots, M) \tag{5.5}$$

$$G_{k,l}^\pm(t) = \sum_{\substack{K \geqslant 0 \\ KM+k-l \geqslant 0}} \binom{KN + n_k^\pm - n_l^\pm}{KM + k - l} t^K \qquad (k, l = 1, \ldots, M) \tag{5.6}$$

Equation (4.1) becomes a set of M coupled linear equations, of the form

$$\sum_{l=1}^{M} G_{k,l}^\pm(t)\, F_l^\pm(t) = G_k^\pm(t) \qquad (k = 1, \ldots, M) \tag{5.7}$$

The solution of this linear system yields the functions $F_k^\pm(t)$, which are generating functions for the integers $A_k^\pm$.

So far t is a formal expansion variable. There is however a natural choice for it, namely

$$t = p^M q^{\tilde{M}} = p^M (1-p)^{N-M} = z(p)^M \tag{5.8}$$

so that the survival probabilities $F^\pm$ can be expressed in terms of the functions $F_k^\pm(t)$. Indeed, with the definitions (5.8) of the variable t and (5.4) of the functions $F_k^\pm(t)$, the probability flow equation (3.15) can be recast as

$$F^\pm = p - \sum_{k=1}^{M} p^k q^{n_k^\pm - k} F_k^\pm(t) \tag{5.9}$$

We now show that the functions $G_k^{\pm}(t)$ and $G_{k,l}^{\pm}(t)$ are algebraic functions of t. These functions are special cases of the functions

$$H_{ij}(t) = \sum_{K\geqslant 0} \binom{KN+j}{KM+i} t^K \tag{5.10}$$

for i, j non-negative integers. Our starting point is the following contour integral representation of the binomial coefficient:

$$\binom{I}{J} = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{(1+z)^I}{z^{J+1}} = \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{1}{u^{J+1}(1-u)^{I-J+1}} \tag{5.11}$$

where the integral is along a small contour around the origin. The second expression is obtained by means of the change of variable $z = u/(1-u)$. Applying Eq. (5.11) to $(I, J) = (KN+j, KM+i)$, and summing over K, we get

$$H_{ij}(t) = \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{u^{M-i-1}(1-u)^{N-M+i-j-1}}{u^M(1-u)^{N-M} - t} \tag{5.12}$$

Now, we need to know which roots of the denominator are inside the integration contour. The geometric sum over K is convergent for $|t| < |u^M(1-u)^{N-M}|$. On the other hand, the polynomial equation

$$u^M(1-u)^{N-M} = t \tag{5.13}$$

the M "small" roots u_α, with $\alpha = 1, \ldots, M$, such that

$$u_\alpha \approx t^{1/M} \omega^{\alpha-1} \qquad (t \to 0) \tag{5.14}$$

with

$$\omega = \mathrm{e}^{2\pi\mathrm{i}/M} \tag{5.15}$$

The small roots can be continued in t. For t real in the range $0 \leqslant t \leqslant t_c$, with

$$t_c = p_c^M q_c^{N-M} = \frac{M^M(N-M)^{N-M}}{N^N} \tag{5.16}$$

the M small roots remain inside the circle $|u| = p_c$, while the other $N-M$ roots of Eq. (5.13), the "large" ones, lie inside the circle $|1-u| = q_c$. The two circles in the u-plane are tangent to each other at $u = p_c$. For $p < p_c$,

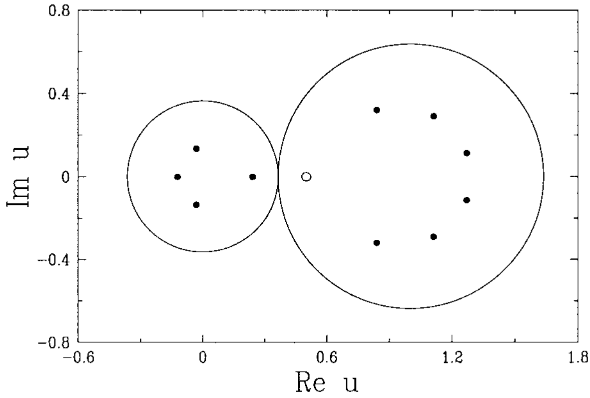

Fig. 9. Roots of the polynomial equation (5.13) for $M=4$, $N=11$, and $p=1/2$.

one of the small roots is p itself, while for $p>p_c$, one of the large roots is p itself. Figure 9 shows a plot of the roots of Eq. (5.13), for $M=4$, $N=11$, and $p=1/2>p_c=4/11$, i.e., $t=1/2048$, while $t_c=108/3125$. The root $p=1/2$ is shown as a large empty symbol.

For $|t|$ small enough, the contour of Eq. (5.12) contains exactly the M small roots u_α, so that we have

$$H_{ij}(t)=\sum_{\alpha=1}^{M}\frac{1}{(M-Nu_\alpha)\,u_\alpha^i(1-u_\alpha)^{j-i}} \tag{5.17}$$

Each of the $u_\alpha(t)$ is an algebraic function of degree N, since it obeys Eq. (5.13). As a consequence, $H_{ij}(t)$ is an algebraic function of degree $\binom{N}{M}$, for any non-negative integers i, j. Indeed, choosing one branch of the function $H_{ij}(t)$ amounts to choosing M branches of $u_\alpha(t)$ among N.

The algebraic function $H_{ij}(t)$ is singular when two roots of the polynomial equation (5.13) coincide, in such a way that the contour of Eq. (5.12) gets pinched. The only non-trivial singularity corresponds to $t=t_c$, where the real positive small root $u=u_1$ merges with the root $u=p$ into a double root at $u=p_c$. In the vicinity of this point, we introduce the notation

$$p=p_c+\delta p, \qquad t\approx t_c\left(1-\frac{N}{2p_cq_c}(\delta p)^2\right) \tag{5.18}$$

For $\delta p>0$, the first root u_1 has a correction in δp, while the other $M-1$ roots are regular in t, so that their corrections are of order $(\delta p)^2$:

$$u_1\approx p_c-\delta p, \qquad u_\alpha=(u_\alpha)_c+\mathcal{O}((\delta p)^2) \qquad (\alpha=2,\ldots,M) \tag{5.19}$$

We thus have $M - Nu_1 \approx N\,\delta p$, hence the estimate

$$H_{ij}(t) \approx \frac{1}{p_c^i q_c^{j-i} N\,\delta p} \approx \frac{1}{p_c^i q_c^{j-i}} \left(\frac{t_c}{2Np_c q_c (t_c - t)} \right)^{1/2} \tag{5.20}$$

This inverse-square-root singularity can alternatively be obtained by means of Eq. (2.20), from the large-K behavior of the coefficients of the series (5.10).

We have thus shown that the functions $G_k^{\pm}(t)$ and $G_{k,l}^{\pm}(t)$ which enter the linear equations (5.7) are algebraic in t, with degree $\binom{N}{M}$. They possess branch points of the form (5.20) at the critical point ($t = t_c$). Unfortunately, these properties are not sufficient to evaluate the critical behavior of the functions $F_k^{\pm}(t)$ near t_c. Indeed, it can be checked that the linear system (5.7) becomes very singular at $t = t_c$: the most singular part of the matrix $G_{k,l}^{\pm}(t)$ as $t \to t_c$ is a matrix of rank one.

It is worth noting that the small roots of the polynomial equation (5.13) also play an important role in recent works[8] devoted to large deviations in exclusion models.

5.2. Probability Flow and Algebraic Trick

An algebraic treatment can also be applied to Eq. (5.9). Consider a fixed $p > p_c$, so that $t < t_c$. If in Eq. (5.9) we replace p by any of the small roots u_α of Eq. (5.13), the r.h.s. vanishes. Indeed this expression has been shown below Eq. (3.18) to vanish for $|z(p)| < z(p_c)$, and we have indeed $|z(u_\alpha)| = |z(p)| < z(p_c)$. Hence

$$u_\alpha = \sum_{k=1}^{M} u_\alpha^k (1 - u_\alpha)^{n_k^{\pm} - k} F_k^{\pm}(t) \qquad (\alpha = 1,..., M) \tag{5.21}$$

This linear system of M equations determines the $F_k^{\pm}(t)$. Equation (5.9) then leads to the survival probabilities $F^{\pm}$.

This algebraic trick is quite powerful for numerical purposes. Indeed, for given p, it only involves solving the polynomial equation (5.13) and the linear system (5.21). Figure 10 shows a plot of the survival probabilities $F^{\pm}$ against the slope v, for different values of p, indicated on the curves. For given p, the algebraic trick has been applied numerically for all rational slopes having a denominator $N \leqslant 40$. The numerical values of $F^{\pm}$ are thus obtained. For a rational slope v, F^+ (respectively, F^-) can be read as the ordinate just after (respectively, just before) the discontinuity of the curve $F(v)$.

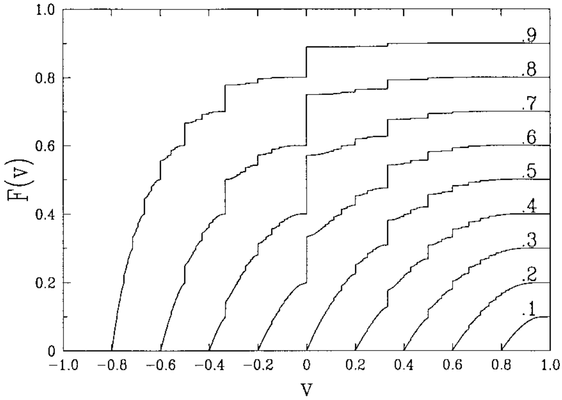

Fig. 10. Plot of the survival probabilities $F^{\pm}$ against v, for several values of p, indicated on the curves. The apparent $v \to 1$ limit of $F(v)$ is $F^{-}(1) = p$ [see Eq. (7.5)], whereas the jump at $v = 1$ to $F^{+}(1) = 1$ is not visible.

The algebraic trick also demonstrates that the $F_k^{\pm}$ are generically algebraic functions of degree $\deg_t(M, N)$ in t, while $F^{\pm}$ are generically algebraic functions of degree $\deg_p(M, N)$ in p, with

$$\deg_t(M, N) = \binom{N}{M}, \qquad \deg_p(M, N) = \binom{N-1}{M} \tag{5.22}$$

Elimination theory can be used to write down explicitly the algebraic relations between p or t and the $F_k^{\pm}$ or $F^{\pm}$. Various examples will be given in Section 7.

If we are interested only in $F^{\pm}$, there is an alternative route.[3] Instead of computing the $F_k^{\pm}$, we can solve a transposed linear system, whose meaning is the following. Suppose that p is replaced by one of the small roots u_α to count the weight of the walks. Then the crossing edges that end on the wall get the correct weights, while the other ones are slightly wrong, because $n_k^{\pm}/k$ is only approximately equal to N/M. By taking a linear combination of the weights corresponding to the u_α, we can give the right weight to any crossing edge. Indeed, given the u_α, we can determine numbers $D_\alpha^{\pm}$ such that we have, for any $k \geqslant 1$

$$\sum_{\alpha=1}^{M} D_\alpha^{\pm} u_\alpha^k (1 - u_\alpha)^{n_k^{\pm} - k} = p^k q^{n_k^{\pm} - k} \tag{5.23}$$

[3] This approach was suggested to us by V. Lafforgue.

This is possible because Eq. (5.23) has to be imposed only for $k=1,..., M$, as a consequence of Eq. (5.2). We can substitute Eq. (5.23) into Eq. (3.15), and use again the fact that the r.h.s. of Eq. (3.15) vanishes when we substitute u_α for p, obtaining thus

$$F^{\pm}=p-\sum_{\alpha=1}^{M} D_\alpha^{\pm} u_\alpha \tag{5.24}$$

We close up this section by investigating the relationship between F^+ and F^-. Using Eq. (5.3), we can recast Eq. (5.9) as

$$\begin{aligned} F^+ &= p(1+tF_M^+)-\sum_{k=1}^{M-1} p^k q^{n_k^{\pm}-k} F_k^+ - tF_M^+ \\ F^- &= p-\sum_{k=1}^{M-1} p^k q^{n_k^{\pm}-k} F_k^- - tF_M^- \end{aligned} \tag{5.25}$$

Since the above equations determine the $F_k^{\pm}$, a comparison between both equations yields

$$F^- = \frac{F^+}{1+tF_M^+}, \qquad F_k^- = \frac{F_k^+}{1+tF_M^+} \qquad (k=1,..., M) \tag{5.26}$$

or equivalently

$$F^+ = \frac{F^-}{1-tF_M^-}, \qquad F_k^+ = \frac{F_k^-}{1-tF_M^-} \qquad (k=1,..., M) \tag{5.27}$$

Finally, Eqs. (3.8) and (3.7) respectively become

$$Q(v)=tF_M^- \tag{5.28}$$

and

$$\Pi(v)=\frac{tF_M^+F^+}{1+tF_M^+}=\frac{tF_M^-F^-}{1-tF_M^+}=tF_M^-F^+=tF_M^+F^- \tag{5.29}$$

5.3. Duality

We now turn to an investigation of the relationship between the survival probability associated with the rational slopes v and $-v$, by exploring the duality approach of Section 4.3 in the rational case. We notice that substituting v for $-v$ amounts to exchanging the roles of the integers M and $\tilde{M}$ in the definitions (5.1).

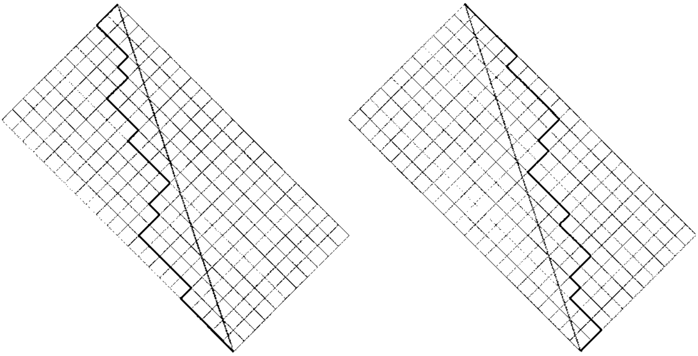

Fig. 11. Combinatorial proof of the identity (5.30).

We start with the following combinatorial observation. Let $(t=n, x=n-2k)$ be a point with integer coordinates on the wall. The number of finite walks starting at the origin, remaining on the same side of the wall and ending at (t, x) is the same for left walks and for right walks. There is indeed an obvious one to one correspondence between both sets of walks, depicted in Fig. 11. The second walk (contributing to right walks) is obtained from the first one (contributing to left walks) by a rotation of angle π around the mid-point $(t/2, x/2)$.

This observation implies the following identity between generating functions

$$F^{\pm}_{M}(t, v)=F^{\pm}_{\tilde{M}}(t, -v) \tag{5.30}$$

In particular, the probability Q that a walk crosses the wall and does so for the first time at a point with integer coordinates, as given by Eq. (5.28), is the same for right and left walks. We shall not use this information for a while, but rather derive many other identities, including Eq. (5.30), by employing only algebraic means.

Following the same strategy as in the irrational case, we start from Eqs. (4.14), (4.15), which now read

$$p-\sum_{k=1}^{M} p^k q^{n^{\pm}_k-k} F^{\pm}_k(t)=0 \qquad (0\leqslant p\leqslant p_c) \tag{5.31}$$

$$p-\sum_{k=1}^{\tilde{M}} p^k q^{\tilde{n}^{\pm}_k-k} \tilde{F}^{\pm}_k(\tilde{t})=0 \qquad (0\leqslant p\leqslant q_c) \tag{5.32}$$

with $\tilde{t}=p^{\tilde{M}}q^{M}$. If we now substitute $q=1-p$ for p in Eq. (5.32) (this turns $\tilde{t}$ into t), and then multiply it by Eq. (5.31), we get

$$\left(1-q-\sum_{k=1}^{M} p^k q^{n_k^\pm-k} F_k^\pm\right)\left(q-\sum_{k=1}^{\tilde{M}} p^k q^{\tilde{n}_k^\pm-k} \tilde{F}_k^\pm\right)=0 \qquad (0\leqslant p\leqslant 1) \tag{5.33}$$

We again use the canonical walk, but now only a finite piece of it. For definiteness, we concentrate our attention on the F_k^+, for both slopes v and $-v$. We first observe that $n_M^+=\tilde{n}_{\tilde{M}}^+=N+1$, so that, if we define

$$F_0^+=1+tF_M^+, \qquad \tilde{F}_0^+=1+t\tilde{F}_{\tilde{M}}^+ \tag{5.34}$$

Eq. (5.33) becomes

$$\left(1-qF_0^+-\sum_{k=1}^{M-1} p^k q^{n_k^+-k} F_k^+\right) \times\left(1-p\tilde{F}_0^+-\sum_{k=1}^{\tilde{M}-1} q^k p^{\tilde{n}_k^+-k} \tilde{F}_k^+\right)=0 \qquad (0\leqslant p\leqslant 1) \tag{5.35}$$

We now change notation for the unknowns inside the sums, just as in the irrational case, labeling the F^+ functions by the value of n_k or $\tilde{n}_k$, rather than with k. We employ the notations (4.18)–(4.23), albeit with the sets $I^\pm$ now being a partition of the integers $l=2,..., N-1$. Eq. (5.35) reads

$$\left(1-qF_0^+-\sum_{l\in I^+} X_l\bar{F}_l^+\right)\left(1-p\tilde{F}_0^+-\sum_{l\in I^-} X_l\bar{F}_l^+\right)=0 \qquad (0\leqslant p\leqslant 1) \tag{5.36}$$

The strategy for expanding this product is the following. We substitute t for $p^M(1-p)^{N-M}$ as often as we can. For fixed t, we thus end up with a polynomial in p of degree less than N. Such a polynomial has to vanish identically, because it has N roots. Indeed, the first factor in Eq. (5.35) vanishes when p is equal to any of the M small roots u_α of the polynomial equation (5.13), while the second factor vanishes on the other $\tilde{M}$ roots.

Now consider $l\in I^+$ and $l'\in I^-$. If $l+l'<N$, then $X_lX_{l'}=X_{l+l'}$. If $l+l'=N$, then $X_lX_{l'}=t$. If $l+l'>N+1$, then $X_lX_{l'}=tX_{l+l'-N}$. The equality $l+l'=N+1$ never occurs. We also have $pX_l=X_{l+1}$ if $l+1<N$ and $pX_{N-1}=t$ if $N-1\in I^+$, and symmetrically $qX_{l'}=X_{l'+1}$ if $l'+1<N$ and $qX_{n-1}=t$ if $N-1\in I^-$. Finally $pq=p(1-p)=X_2$. All these properties can be checked by combining the inequalities defining k_l and $k_{l'}$ [see Eq. (4.18)]. All the X_l's have different degrees ranging from 2 to $N-1$, so that, together with any two of the three polynomials 1, p and $1-p$, they

form a basis of the polynomials of degree less than N. We concentrate for a while on terms of degree less than two in p. We are left with the sum of a function of t and of $-(1-p)\,\tilde{F}_0^+ - pF_0^+$, which has to vanish identically. This implies in particular $\tilde{F}_0^+ = F_0^+$, which is just the identity (5.30), that we have proved in the beginning of this section by combinatorial means. We set

$$\bar{F}_1^+ = \tilde{F}_0^+ = F_0^+ \tag{5.37}$$

and $I_0^+ = I^+ \cup \{1\}$ and $I_0^- = I^- \cup \{1\}$. Just as in the irrational case, we meet many simplifications, and we are left with

$$1 - \bar{F}_1^+ - \sum_{l=2}^{N-1} \bar{F}_l^+ X_l + \sum_{l \in I_0^+,\, l' \in I_0^-} \bar{F}_l^+ \bar{F}_{l'}^+ t^{\rho(l,\, l')} X_{l+l'-N\rho(l,\, l')} = 0 \tag{5.38}$$

with the convention $X_0 = 1$, and with

$$\rho(l, l') = \begin{cases} 0 & \text{if } \; l + l' < N \\ 1 & \text{if } \; l + l' \geqslant N \end{cases} \tag{5.39}$$

A term-by-term identification then leads to

$$\bar{F}_1^+ = 1 + t \sum_{\substack{l \in I_0^-,\, l' \in I_0^+ \\ l + l' = N}} \bar{F}_l^+ \bar{F}_{l'}^+ \tag{5.40}$$

and

$$\bar{F}_l^+ = \sum_{\substack{l' \in I_0^-,\, l'' \in I_0^+ \\ l' + l'' = l \bmod N}} \bar{F}_{l'}^+ \bar{F}_{l''}^+ t^{\rho(l',\, l'')} \qquad (l = 2,..., N-1) \tag{5.41}$$

Equations (5.40) and (5.41) provide $N-1$ equations for the $N-1$ unknowns $\bar{F}_l^+$. When t goes to 0, Eq. (5.40) becomes trivial, while the other ones become identical to the recursion relation (4.26). As any irrational slope v can be approximated by a rational one that leads to the same coefficients $\bar{A}_l$ up to an arbitrary given l, we thus obtain a proof of the third recursion relation (4.26) by algebraic means, without any recourse to analysis.

6. CRITICAL BEHAVIOR

This section is devoted to the analysis of the "critical behavior" as $p \to p_c$ of the survival probabilities $F^\pm$ and of related quantities in the convergent regime ($p > p_c$), for a fixed slope v, either rational or not. This investigation also yields quantitative predictions concerning the amplitudes $C^{\pm}(v)$ in the marginal regime ($p = p_c$) and the prefactors $b^{\pm}(np_c)$ in the large-deviation regime ($p < p_c$).

6.1. Convergent Regime

We consider first the convergent regime, for a rational slope. The behavior (4.12) of the integers $A_k^{\pm}$ implies that the generating series $F_k^{\pm}(t)$ have square-root singularities at $t = t_c$, as shown by Eq. (2.20). We again employ the notations (5.18), and we define the positive amplitudes $\Phi_k^{\pm}$ such that

$$F_k^{\pm}(t) \approx (F_k^{\pm})_c - \Phi_k^{\pm}\, |\delta p| \qquad (k = 1,..., M,\ \delta p \to 0) \tag{6.1}$$

with $(F_k^{\pm})_c = F_k^{\pm}(t_c)$. Similarly, the survival probabilities $F^{\pm}$ are expected to vanish linearly as $p \to p_c + 0$, with amplitudes $\Phi^{\pm}$:

$$F^{\pm} \approx \Phi^{\pm}\, \delta p \qquad (\delta p \to +0) \tag{6.2}$$

The critical amplitudes $\Phi^{\pm}$ and $\Phi_k^{\pm}$ can be determined by means of the algebraic trick as follows. By differentiating Eq. (5.9), we obtain

$$\frac{\mathrm{d}F^{\pm}}{\mathrm{d}p} = 1 - \sum_{k=1}^{M} p^k q^{n_k^{\pm} - k} \left\{ \left(\frac{k}{p} - \frac{n_k^{\pm} - k}{q} \right) F_k^{\pm} + \frac{\mathrm{d}F_k^{\pm}}{\mathrm{d}p} \right\} \tag{6.3}$$

Now, by inserting the limits

$$\begin{aligned} &\frac{\mathrm{d}F^{\pm}}{\mathrm{d}p} \to \Phi^{\pm}, \qquad \frac{\mathrm{d}F_k^{\pm}}{\mathrm{d}p} \to -\Phi_k^{\pm} \qquad (p \to p_c + 0) \\ &\frac{\mathrm{d}F^{\pm}}{\mathrm{d}p} \to 0, \qquad \frac{\mathrm{d}F_k^{\pm}}{\mathrm{d}p} \to \Phi_k^{\pm} \qquad (p \to p_c - 0) \end{aligned} \tag{6.4}$$

into Eq. (6.3), and using Eq. (3.10), we obtain two different expressions for the amplitudes $\Phi^{\pm}$:

$$\begin{aligned} \Phi^{\pm} &= 2 \left(1 + \sum_{k=1}^{M} p_c^k q_c^{n_k^{\pm} - k - 1} (1 - \mathrm{Frac}^{\mp}(k/p_c)) (F_k^{\pm})_c \right) \\ &= 2 \sum_{k=1}^{M} p_c^k q_c^{n_k^{\pm} - k} \Phi_k^{\pm} \end{aligned} \tag{6.5}$$

The middle side of Eq. (6.5) gives $\Phi^{\pm}$ in terms of the $(F_k^{\pm})_c$, which can be obtained by the algebraic trick at $p = p_c$. The rightmost side of Eq. (6.5) then allows to determine the $\Phi_k^{\pm}$ by means of a second use of the algebraic trick. Indeed the behavior (5.19) of the roots u_α implies $\mathrm{d}u_\alpha/\mathrm{d}p \to -\delta_{\alpha,1}$ in the $p \to p_c + 0$ limit. As a consequence, the $\phi_k^{\pm}$ can be determined from the following linear system of M equations

$$\frac{\delta_{\alpha,1}}{2}\Phi^{\pm} = \sum_{k=1}^{M} (u_\alpha)_c^k (1-(u_\alpha)_c)^{n_k^{\pm}-k}\, \Phi_k^{\pm} \qquad (\alpha = 1,..., M) \tag{6.6}$$

It turns out that the amplitudes $\Phi^{\pm}$ and $\Phi_k^{\pm}$ govern the behavior of various quantities for a rational slope. First, Eq. (2.20) yields the asymptotic behavior of the integers $A_k^{\pm}$ for large k. We thus recover Eq. (4.12), and we obtain expressions for the amplitudes $B_k^{\pm}$, which asymptotically only depend on k modulo M:

$$B_{KM+k}^{\pm} \approx N p_c q_c (t_c)^{k/M}\, \Phi_k^{\pm} \qquad (k = 1,..., M,\ K \gg 1) \tag{6.7}$$

This result can be recast as follows. Since $p_c = M/N$ is rational, the fractional part $\mathrm{Frac}^{\pm}(k/p_c) = \mathrm{Frac}^{\pm}(kN/M)$ takes M rational values with denominator M, and only depends on k modulo M. Conversely, any function of k modulo M can be considered as a function of $\mathrm{Frac}^{\pm}(k/p_c)$. Hence Eq. (6.7) is equivalent to

$$B_k^{\pm} \approx B^{\pm}(k/p_c) \tag{6.8}$$

i.e.,

$$A_k^{\pm} \approx B^{\pm}(k/p_c)\, \frac{p_c}{(2\pi q_c k^3)^{1/2}}\, (p_c q_c^{q_c/p_c})^{-k} \qquad (k \gg 1) \tag{6.9}$$

where $B^{\pm}(x)$ are periodic functions, with unit period. Since the functional form of Eq. (6.9) holds independently of the slope v, provided v is rational, we make the reasonable hypothesis that the law (6.9) also holds when v is irrational, albeit with one single periodic function $B(x)$.

The bilateral sequence of integers $\bar{A}_n^{\pm}$, introduced in Eq. (4.21), can be argued to exhibit an asymptotic behavior analogous to Eq. (6.9):

$$\bar{A}_n^{\pm} \approx \beta^{\pm}(\theta_n)\, \frac{1}{(2\pi p_c q_c n^3)^{1/2}}\, (p_c^{p_c} q_c^{q_c})^{-n} \qquad (n \gg 1) \tag{6.10}$$

where the arguments θ_n of the periodic functions $\beta^{\pm}(\theta)$ have been defined in Eq. (4.18). The estimate (6.10), with its periodic functions $\beta^{\pm}(\theta)$,

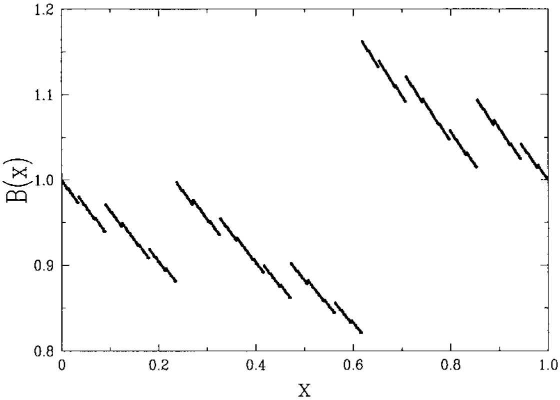

Fig. 12. Plot of the periodic amplitude $B(x)$, for the golden slope.

involves p_c and q_c in a symmetric fashion, contrary to Eq. (6.9). Furthermore it encodes simultaneously the asymptotic behavior of the sequences $A_k^{\pm}$ associated with both slopes $\pm v$. These are advantages of the duality approach. Eq. (4.19) also implies the following relationship

$$B^{\pm}(x) = (p_c q_c^{q_c/p_c})^{-\theta} \beta^{\pm}(\theta), \qquad \theta = p_c(1-x) \qquad (0 \leqslant \theta \leqslant p_c) \tag{6.11}$$

between the periodic functions of the estimates (6.9) and (6.10), with corresponding arguments. Figures 12 and 13 respectively show plots of the periodic functions $B(x)$ and $\beta(\theta)$, again for the golden slope given in Eq. (2.23).

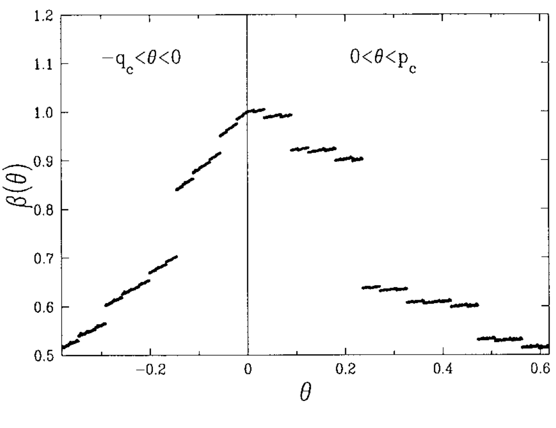

Fig. 13. Plot of the periodic amplitude $\beta(\theta)$, for the golden slope.

6.2. Marginal and Large-Deviation Regimes

Quantities pertaining to the marginal regime ($p = p_c$) and to the large-deviation regime ($p < p_c$) can also be characterized in terms of the critical amplitudes $\Phi^{\pm}$ and $\Phi_k^{\pm}$, for a rational slope.

In these two regimes, the limit survival probabilities $F^{\pm}$ vanish, so that the probability flow equation (3.14) becomes

$$F^{\pm}(n, v) = \sum_{k \geqslant \mathrm{Int}^{\pm}(np_c)+1} A_k^{\pm} p^k q^{n_k^{\pm}-k} \tag{6.12}$$

In the marginal regime ($p = p_c$), the behavior of $F^{\pm}(n, v)$ for large n can be derived by inserting the estimates (4.12) and (6.7) into Eq. (6.12). The terms in the r.h.s. of that equation decay as the power law $k^{-3/2}$, so that the result is rather insensitive to the details of the sequences $A_k^{\pm}$. Indeed the amplitudes $\Phi_k^{\pm}$ only appear through the combinations which enter the rightmost side of Eq. (6.5). we thus recover the general result (2.29), with an amplitude

$$C^{\pm}(v) = \left(\frac{p_c q_c}{2}\right)^{1/2} \Phi^{\pm} = \left(\frac{1-v^2}{8}\right)^{1/2} \Phi^{\pm} \tag{6.13}$$

in agreement with Eq. (2.35).

The algebraic trick at the critical point thus allows a numerical determination of the amplitude $C(v)$ of the law (2.29). Figure 14 shows a logarithmic plot of this amplitude, obtained from data at rational slopes, along the lines of Fig. 10. The curve is centrally symmetric, as a consequence of the duality symmetry (2.30). Figure 15 shows a similar, albeit more appealing, plot of the combination

$$y^{\pm}(v) = \ln C^{\pm}(v) + \frac{1}{2} \ln \frac{q_c}{p_c} = \ln C^{\pm}(v) + \frac{1}{2} \ln \frac{1+v}{1-v} \tag{6.14}$$

This quantity obeys the following bounds, shown as dashed lines in Fig. 15:

$$\begin{gathered}
-\frac{\ln 2}{2} \leqslant y^{\pm}(v) \leqslant \frac{\ln 2}{2} - \ln(1-v) \qquad (-1 \leqslant v \leqslant 0) \\
\ln(1+v) - \frac{\ln 2}{2} \leqslant y^{\pm}(v) \leqslant \frac{\ln 2}{2} \qquad (0 \leqslant v \leqslant 1)
\end{gathered} \tag{6.15}$$

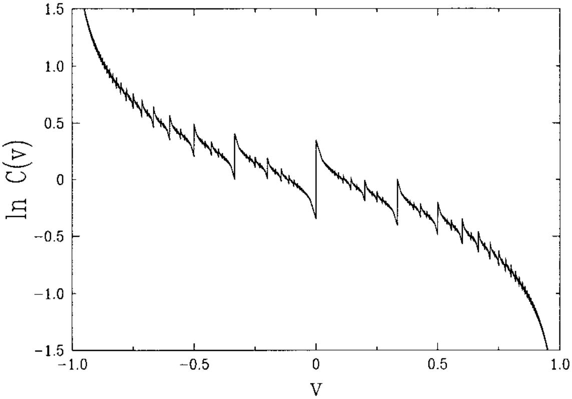

Fig. 14. Logarithmic plot of the critical amplitude $C(v)$ against v.

The results (7.12) and (7.27), to be derived later, show that the bounds (6.15) are saturated by the slopes with either $M=1$ or $\tilde{M}=1$. These bounds lead to the limit values

$$y(\pm 1) = \pm\frac{\ln 2}{2} \tag{6.16}$$

The investigation of the large-deviation regime ($p<p_c$) is more involved. The sum in the r.h.s. of Eq. (6.12) is now exponentially convergent, so that it is dominated by the fine structure of the $A_k^{\pm}$ near the

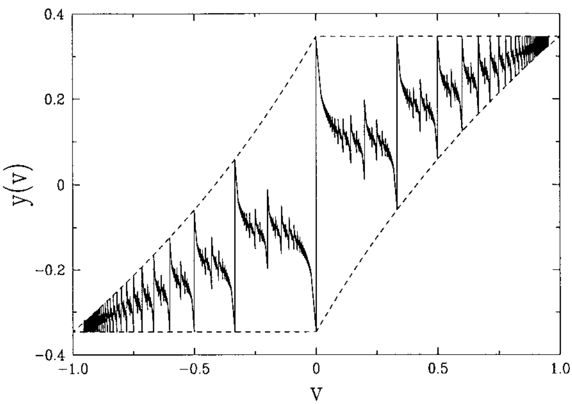

Fig. 15. Plot of $y(v)$, defined in Eq. (6.14), against v.

lower bound, i.e., for $k \approx np_c$. We recover after some manipulations the general result (2.21), with amplitudes which only depend on n modulo N:

$$b^{\pm}_{KN+n} \approx \frac{Nq_c}{p_c^{1/2}(1-\mathrm{e}^{-NS(v)})} \times \sum_{k=1}^{M} q^{1-\mathrm{Frac}^{\mp}(k/p_c)} \mathrm{e}^{[(n-k/p_c)-N\Theta(n-k/p_c)]\,S(v)} (t_c)^{k/M}\, \Phi_k^{\pm} \tag{6.17}$$

for $n = 1,..., N$ and $K \gg 1$, where Θ is the Heaviside step function, and $S(v)$ the entropy function of Eq. (2.13). This result agrees with the existence of a periodic prefactor to the law (2.21), since any function of n modulo N can be viewed as a function of $\mathrm{Frac}^{\pm}(np_c) = \mathrm{Frac}^{\pm}(nM/N)$.

7. FURTHER RESULTS IN SPECIFIC CASES

7.1. The Limit Case $v = 1$, i.e., $p_c = 0$

In this limit situation quantities with the superscript $+$ are equal to the same quantities for the whole unconstrained set of random walks. Quantities with the superscript $-$ are only constrained by the condition that the first step is to the left ($\varepsilon_1 = -1$).

The general formalism of this paper agrees with these considerations. Indeed, any value of $p > 0$ corresponds to the convergent regime. Equation (2.7) implies that the one-time distribution functions read

$$P^+(n, v) = 1, \qquad P^-(n, v) = 1 - q^n \qquad (n \geqslant 1) \tag{7.1}$$

in agreement with the value $S = -\ln q$ of the entropy function [see Eq. (2.13)]. We have therefore

$$f^+(z, v) = \frac{1}{1-z}, \qquad f^-(z, v) = \frac{1-qz}{1-z} \tag{7.2}$$

so that

$$F^+(n, v) = 1, \qquad F^-(n, v) = p \qquad (n \geqslant 1) \tag{7.3}$$

The integers $\mathcal{N}^{\pm}_{n,k}$ are also remarkably simple:

$$\mathcal{N}^{+}_{n,k} = \binom{n}{k}, \qquad \mathcal{N}^{-}_{n,k} = \binom{n-1}{k-1} \qquad (n \geqslant 1, k \geqslant 1) \tag{7.4}$$

The limit survival probabilities read therefore

$$F^+ = 1, \qquad F^- = p, \qquad \Pi = q \tag{7.5}$$

which can be recovered by elementary considerations: we have $\Pi = \text{Prob}\{V=1\} = \text{Prob}\{\varepsilon_1 = 1\} = q$ and $F^+ = 1$, hence $F^- = F^+ - \Pi = 1 - q = p$.

7.2. The Limit Case $v = -1$, i.e., $p_c = 1$

This situation is the dual of the previous one. Quantities with the superscript $-$ vanish identically, while quantities with the superscript $+$ have the single contribution of the walk with all steps to the left ($\varepsilon_n = -1$ for all n).

Any value of $p < 1$ now corresponds to the large-deviation regime. Equation (2.7) leads to

$$P^+(n, v) = p^n, \qquad P^-(n, v) = 0 \qquad (n \geqslant 1) \tag{7.6}$$

in agreement with the value $S = -\ln p$ of the entropy function. We have therefore

$$f^+(z, v) = \frac{1}{1 - pz}, \qquad f^-(z, v) = 0 \tag{7.7}$$

so that the survival probabilities read

$$F^+(n, v) = p^n, \qquad F^-(n, v) = 0 \qquad (n \geqslant 0) \tag{7.8}$$

The only non-zero integers $\mathcal{N}^{\pm}_{n,k}$ are

$$\mathcal{N}^{+}_{n,n} = 1 \qquad (n \geqslant 0) \tag{7.9}$$

7.3. Rational Slopes with $M = 1$, i.e., $v = 1 - 2/N$ or $p_c = 1/N$

The class of rational slopes with $M = 1$ owes its simplicity to the fact that only one unknown function, $F^{\pm}_1(t)$, is involved in the algebraic trick approach.

For $M = 1$ and $N \geqslant 2$, we have $v = 1 - 2/N$, $p_c = 1/N$, $q_c = (N-1)/N$, $t = pq^{N-1}$ and $t_c = (N-1)^{N-1}/N^N$. In the convergent regime ($p > p_c$), the polynomial equation (5.13) has only one small root, u_1, which is in the interval $0 < u_1 < p_c$. Equation (5.21) has the explicit solutions

$$F_1^- = \frac{u_1}{t}, \qquad F_1^+ = \frac{u_1}{t(1-u_1)} \tag{7.10}$$

We thus obtain

$$F^- = p - u_1, \qquad F^+ = \frac{p-u_1}{1-u_1}, \qquad Q = u_1, \qquad \Pi = \frac{u_1(p-u_1)}{1-u_1} \tag{7.11}$$

We notice that $u_1(t)$ and $F_1^{\pm}(t)$ are algebraic functions of degree N in t, while $u_1(p)$ and $F^{\pm}(p)$ are algebraic functions of degree $N-1$ in p, in agreement with Eq. (5.22).

The critical behavior of the quantities given in Eqs. (7.10), (7.11) is characterized by the amplitudes

$$\begin{gathered}
(F_1^-)_c = \left(\frac{N}{N-1}\right)^{N-1}, \qquad (F_1^+)_c = \left(\frac{N}{N-1}\right)^{N} \\
\Phi_1^- = \frac{N^N}{(N-1)^{N-1}}, \qquad \Phi_1^+ = \frac{N^{N+2}}{(N-1)^{N+1}} \\
\Phi^- = 2, \qquad \Phi^+ = \frac{2N}{N-1}, \qquad C^- = \frac{(2(N-1))^{1/2}}{N}, \qquad C^+ = \left(\frac{2}{N-1}\right)^{1/2}
\end{gathered} \tag{7.12}$$

The integers $A_k^{\pm}$ can be obtained in several ways. The easiest route consists in evaluating them as contour integrals of the generating series $F_1^{\pm}(t)$. The expressions (7.10) yield

$$\begin{aligned}
A_k^- &= \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{k+1}}\, u_1 = \oint \frac{\mathrm{d}u_1}{2\pi \mathrm{i} u_1^k} \frac{1-Nu_1}{(1-u_1)^{Nk-k+1}} \\
A_k^+ &= \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{k+1}} \frac{u_1}{1-u_1} = \oint \frac{\mathrm{d}u_1}{2\pi \mathrm{i} u_1^k} \frac{1-Nu_1}{(1-u_1)^{Nk-k+2}}
\end{aligned} \tag{7.13}$$

The integrals over t are transformed to integrals over u_1 by means of the identity $t = u_1(1-u_1)^{N-1}$, whereas the latter are linear combinations of the integrals over u which enter Eq. (5.11). We thus obtain the integers $A_k^{\pm}$ as ratios of factorials:

$$A_k^- = \frac{(Nk-2)!}{k!\,(Nk-k-1)!}, \qquad A_k^+ = \frac{(Nk)!}{k!\,(Nk-k+1)!} \tag{7.14}$$

These formulas can also be obtained combinatorially, by means of the Raney principle.[9] They agree with the observation, made below Eq. (4.10), that the A_k^- saturate the lower bound, while the A_k^+ saturate the upper bound. The corresponding amplitudes $B_k^\pm$ read asymptotically

$$B_k^- \approx q_c = \frac{N-1}{N}, \qquad B_k^+ \approx \frac{1}{q_c} = \frac{N}{N-1} \tag{7.15}$$

The situation of an immobile obstacle, i.e., of a vertical wall, with slope $v=0$, corresponds to $M=\tilde{M}=1$ and $N=2$. In this case, we have $p_c=1/2$ and $u_1=1-p$ for $p>1/2$, so that the results (7.10)–(7.11) can be further simplified to

$$\begin{gathered} F_1^- = \frac{1}{p}, \qquad F_1^+ = \frac{1}{p^2}, \qquad F^- = 2p-1, \qquad F^+ = \frac{2p-1}{p} \\ Q = 1-p, \qquad \Pi = \frac{(2p-1)(1-p)}{p} \end{gathered} \tag{7.16}$$

The discontinuity Π is maximal for $p=1/\sqrt{2}$, where it equals

$$\Pi_{\max} = (\sqrt{2}-1)^2 = 0.171573 \tag{7.17}$$

This value is the largest possible discontinuity met in this problem, except the trivial one at $v=1$, namely $\Pi=q$ [see Eq. (7.5)].

This example call be alternatively investigated ab initio. Indeed, the integers $\mathcal{N}^\pm_{n,k}$ which obey Eq. (3.12) can be written as differences of binomial coefficients:

$$\mathcal{N}^-_{n,k} = \binom{n-1}{k-1} - \binom{n-1}{k}, \qquad \mathcal{N}^+_{n,k} = \binom{n}{k} - \binom{n}{k+1} \tag{7.18}$$

Equation (3.13) then yields, in agreement with Eq. (7.14) for $N=2$,

$$\begin{aligned} A_k^- &= \mathcal{N}^-_{2k-1,k} = \binom{2k-2}{k-1} - \binom{2k-2}{k} = \frac{(2k-2)!}{k!\,(k-1)!} = \mathrm{C}_{k-1} \\ A_k^+ &= \mathcal{N}^+_{2k,k} = \binom{2k}{k} - \binom{2k}{k+1} = \frac{(2k)!}{k!\,(k+1)!} = \mathrm{C}_k \end{aligned} \tag{7.19}$$

where C_k are the Catalan numbers.[9]

7.4. Rational Slopes with $\tilde{M}=1$, i.e., $v=-1+2/N$ or $p_c=1-1/N$

The class of walls with $\tilde{M}=1$, i.e., a slope $v=2/N-1$, gives a nice illustration of the duality approach, exposed in Section 5.3.

In this situation, we have $M=N-1$, $p_c=1-1/N$, $q_c=1/N$, $t=p^{N-1}q$, and $t_c=(N-1)^{N-1}/N^N$. According to the general rules of Section 5.3, we introduce $N-1$ functions $\bar{F}_1^+,..., \bar{F}_{N-1}^+$. The function $\bar{F}_1^+$ pertains to both v and $-v$, while all the other ones only pertain to v. We have $I_0^-=\{1\}$ and $I_0^+=\{1,..., N-1\}$, so that the duality equations (5.40), (5.41) respectively read

$$\begin{aligned}\bar{F}_1^+&=1+t\bar{F}_1^+\bar{F}_{N-1}^+\\ \bar{F}_k^+&=\bar{F}_1^+\bar{F}_{k-1}^+\qquad (k=2,..., N-1)\end{aligned}\tag{7.20}$$

So, we have $\bar{F}_k^+=(\bar{F}_1^+)^k$ for $k=1,..., N-1$, and

$$\bar{F}_1^+=1+t(\bar{F}_1^+)^N\tag{7.21}$$

The solution $\bar{F}_1^+(t)$ is the branch of Eq. (7.21) which is regular at $t=0$. A comparison of Eq. (7.21) with the identity $t=p^{N-1}q$ shows that $\bar{F}_1^+=1/p$. We thus have

$$\bar{F}_k^+=\frac{1}{p^k}\qquad (k=1,..., N-1)\tag{7.22}$$

In order to obtain a complete combinatorial description of the problem, we must recall the relation between the functions $\bar{F}_k^+$ and F_k^+. We have

$$n_k^+=k+1\qquad (k=1,..., N-2),\qquad n_{N-1}^+=N+1\tag{7.23}$$

so that $\bar{F}_1^+=1+tF_m^+=1+t\tilde{F}_1^+$ and $\bar{F}_k^+=F_{k-1}^+$ for $k=2,..., N-1$. We thus obtain the remarkably simple result

$$F_k^+=\frac{1}{p^{k+1}}\qquad (k=1,..., N-1)\tag{7.24}$$

Equations (5.9) and (5.26)–(5.29) then yield

$$F_k^-=\frac{1}{p^k}\qquad (k=1,..., N-1)\tag{7.25}$$

and

$$F^- = N_p - (N-1), \qquad F^+ = \frac{Np-(N-1)}{p}$$
$$Q = q, \qquad \Pi = \frac{q(Np-(N-1))}{p} \tag{7.26}$$

Equations (7.24)–(7.26) show that the $F_k^{\pm}(t)$ are algebraic functions of degree N in t, just as $p(t)$, while $F^{\pm}(p)$ are rational functions, i.e., algebraic functions of degree 1 in p, in agreement with Eq. (5.22). Finally, the results (7.16) can be recovered by setting $N=2$ in Eqs. (7.24)–(7.26).

The simplicity of the results (7.24)–(7.26), especially for F^-, suggests that they can be obtained by elementary means. This is indeed the case.[4] Fix $p > p_c = M/N$. The probability to start to the right and to touch the wall is $q = 1-p$, because if the first step is to the right (probability q), then the walk crosses the wall with probability one. Let R be the probability to start to the left and touch the wall. Then, by definition, $F^- = p - R$. The main point is as follows: a walk touching the wall touches it with probability one at least once at a point with integer coordinates. If the walk starts to the right this is clear, because $\tilde{M} = 1$, so passing through an integer point is the only way to touch the wall coming from the right. A walk starting to the left and touching the wall either does it for the first time at an integer point (and we are done), or it crosses it at some point from left to right, and then with probability one touches the wall again later coming from the right, this time at an integer point, because this is the only way to touch from the right. In particular, R is also the probability to start to the left and touch the wall at an integer point. Now, if we pick an integer point on the wall and compute the probability that a walk starts to the left and passes though this point versus the probability that a walk starts on the right and passes though this point, the independent weight is the same for both, and what remains in the ratio is the quotient of two binomial coefficients, giving M. Although a walk can pass through several integer points, as this ratio is the same for any integer point, this is enough to show that R is exactly M times the probability to start to the right and cross the wall. The latter is q, so $R = Mq$ and $F^- = p - R = P - Mq = Np - M$, in agreement with Eq. (7.26).

To finish the combinatorial analysis, we show that $Q = q$, again in agreement with Eq. (7.26). This is a consequence of an observation made

[4] The following argument is due to V. Lafforgue.

at the beginning of Section 5.3, and summarized in Eq. (5.30). The probability Q that a walk crosses the wall and does it for the first time at a point with integer coordinates is the same for right and left walks. But for right walks $Q=q$, because any walk starting to the right has to cross the wall, and can only do so at an integer point.

The critical behavior of the quantities given in Eqs. (7.24)–(7.26) is characterized by the amplitudes

$$\Phi_k^- = k\left(\frac{N}{N-1}\right)^{k+1}, \qquad \Phi_k^+ = (k+1)\left(\frac{N}{N-1}\right)^{k+2}$$

$$\Phi^- = N, \qquad \Phi^+ = \frac{N^2}{N-1}, \qquad C^- = \left(\frac{N-1}{2}\right)^{1/2}, \qquad C^+ = \frac{N}{(2(N-1))^{1/2}} \tag{7.27}$$

The integers $A^{\pm}_{KM+k}$ can again be evaluated as contour integrals of the generating series $F_k^{\pm}(t)$. In analogy with Eq. (7.13), the expressions (7.24), (7.25) yield

$$\begin{aligned} A^-_{KM+k} &= \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{K+1}}\, \frac{1}{p^k} = \oint \frac{\mathrm{d}p}{2\pi \mathrm{i} p^{KM+k+1}}\, \frac{M-Np}{(1-p)^{K+1}} \\ A^+_{KM+k} &= \oint \frac{\mathrm{d}t}{2\pi \mathrm{i} t^{K+1}}\, \frac{1}{p^{k+1}} = \oint \frac{\mathrm{d}p}{2\pi \mathrm{i} p^{KM+k+2}}\, \frac{M-Np}{(1-p)^{K+1}} \end{aligned} \tag{7.28}$$

(remember that $M=N-1$), and finally

$$A^-_{KM+k} = \frac{k(KN+k-1)!}{K!\,(KM+k)!}, \qquad A^+_{KM+k} = \frac{(k+1)(KN+k)!}{K!\,(KM+k+1)!} \tag{7.29}$$

The corresponding amplitudes $B^{\pm}_{KM+k}$ read asymptotically

$$\begin{aligned} B^-_{KM+k} &\approx B^-(k/M) = k(M+1)^{-k/M} \qquad (k=1,\ldots, M) \\ B^+_{KM+k} &\approx B^+(k/M) = \frac{(k+1)(M+1)^{1-k/M}}{M} \qquad (k=0,\ldots, M-1) \end{aligned} \tag{7.30}$$

These expressions agree with Eq. (6.8), since $k/M = \mathrm{Frac}^{\pm}(k/p_c)$ in both cases. We have in particular $B^-(1) = M/N = p_c$, and $B^+(0) = N/M = 1/p_c$.

For M large, both amplitude functions $B^{\pm}(x)$ exhibit the scaling form

$$B^{\pm}(x) \approx x M^{1-x} \qquad (0 \leqslant x \leqslant 1,\ M \gg 1) \tag{7.31}$$

This expression takes its maximal value, $B^{\pm}_{\max} \approx M/(\mathrm{e} \ln M)$, for $x_{\max} = 1/(\ln M)$. This estimate is only by a factor $\ln M$ below the upper bound of Eq. (4.13), which reads $1/q_c = N = M+1$.

7.5. A Case Study: The Slope $v = 1/5$ or $p_c = 2/5$, i.e., $M = 2$, $\tilde{M} = 3$

In this section we study the simplest case of a rational slope which belongs to neither of the classes studied before, namely $M=2$ and $\tilde{M}=3$. This example corresponds to the slope $v = 1/5$. We have $N=5$, $p_c = 2/5$, and $t_c = 108/3125$. We shall apply successively the three techniques exposed in Section 5.

We start with the approach developed in Section 5.1, based on the continuity of the path.

The functions involved in the linear system (5.7) read

$$G_1^{\pm} = \sum_{k \geqslant 0} \binom{5k+2}{2k} t^k, \qquad G_2^{+} = \sum_{k \geqslant 0} \binom{5k+5}{2k+1} t^k, \qquad G_2^{-} = \sum_{k \geqslant 0} \binom{5k+4}{2k+1} t^k$$

$$G_{11}^{\pm} = G_{22}^{\pm} = \sum_{k \geqslant 0} \binom{5k}{2k} t^k, \qquad G_{12}^{+} = \sum_{k \geqslant 1} \binom{5k-3}{2k-1} t^k \tag{7.32}$$

$$G_{12}^{-} = \sum_{k \geqslant 1} \binom{5k-2}{2k-1} t^k, \qquad G_{21}^{+} = \sum_{k \geqslant 0} \binom{5k+3}{2k+1} t^k, \qquad G_{21}^{-} = \sum_{k \geqslant 0} \binom{5k+2}{2k+1} t^k$$

The solution of this system leads to the generating series $F_k^{\pm}(t)$ of Eq. (5.4) in the form

$$F_1^{\pm} = \frac{G_1^{\pm} G_{22}^{\pm} - G_2^{\pm} G_{12}^{\pm}}{G_{11}^{\pm} G_{22}^{\pm} - G_{21}^{\pm} G_{12}^{\pm}}, \qquad F_2^{\pm} = \frac{G_2^{\pm} G_{11}^{\pm} - G_1^{\pm} G_{21}^{\pm}}{G_{11}^{\pm} G_{22}^{\pm} - G_{21}^{\pm} G_{12}^{\pm}} \tag{7.33}$$

The Taylor expansions of these quantities lead to the integers $A_k^{\pm}$, namely

$$\begin{aligned} F_1^{+} &= 1 + 7t + 99t^2 + 1768t^3 + 35530t^4 + 766935t^5 \\ &\quad + 17368680t^6 + 407139120t^7 \\ &\quad + 9794689506t^8 + 240455164510t^9 + 5997744185435t^{10} + \cdots \\ F_2^{+} &= 2 + 23t + 377t^2 + 7229t^3 + 151491t^4 + 3361598t^5 + 77635093t^6 \\ &\quad + 1846620581t^7 + 44930294909t^8 + 1113015378438t^9 \\ &\quad + 27976770344941t^{10} + \cdots \end{aligned}$$

$$F_1^- = 1 + 5t + 66t^2 + 1144t^3 + 22610t^4 + 482885t^5 + 10855425t^6$$
$$+ 253086480t^7 + 6063379218t^8 + 148365952570t^9$$
$$+ 3692150267960t^{10} + \cdots$$
$$F_2^- = 2 + 19t + 293t^2 + 5452t^3 + 112227t^4 + 2460954t^5 + 56356938t^6$$
$$+ 1332055265t^7 + 32251721089t^8 + 795815587214t^9$$
$$+ 19939653287183t^{10} + \cdots \qquad (7.34)$$

An arbitrary number of terms can be evaluated with the help of a computer. We have, however, found no simple closed form for the coefficients $A_k^\pm$, such as Eq. (7.14) for $M = 1$ or Eq. (7.29) for $\widetilde{M} = 1$.

We now turn to the algebraic trick of Section 5.2. For $0 < t < t_c$, the polynomial equation (5.13), i.e., $u^2(1-u)^3 = t$, has two small solutions u_1 and u_2, such that $u_2 < 0 < u_1 < 2/5$. The relation between t and any of the $F_k^\pm$ can be obtained by means of an algebraic elimination of u_1 and u_2 between Eqs. (5.13) and (5.21), a tedious task that we prefer to leave to the computer. We thus obtain

$$0 = t^6(F_1^+)^{10} - t^7(F_1^+)^7 - 11t^3(F_1^+)^5 - t^2(F_1^+)^4 - 7t(F_1^+)^2 + F_1^+ - 1$$
$$0 = t^{11}(F_2^+)^{10} + 10t^{10}(F_2^+)^9 + 45t^9(F_2^+)^8 + t^7(120t+1)(F_2^+)^7$$
$$+ 6t^6(35t+1)(F_2^+)^6 + t^5(252t+17)(F_2^+)^5 + 30t^4(7t+1)(F_2^+)^4$$
$$+ 5t^3(24t+7)(F_2^+)^3 + t^2(45t+26)(F_2^+)^2 + (10t^2+11t-1)\,F_2^+ + t + 2$$
$$0 = t^4(F_1^-)^{10} - 3t^3(F_1^-)^8 + 3t^2(F_1^-)^6 + 11t^2(F_1^-)^5 - t(F_1^-)^4$$
$$- 4t(F_1^-)^3 + F_1^- - 1$$
$$0 = t^9(F_2^-)^{10} - 9t^8(F_2^-)^9 + 36t^7(F_2^-)^8 - 84t^6(F_2^-)^7 + 126t^5(F_2^-)^6$$
$$- 2t^4(t+63)(F_2^-)^5 + 3t^3(3t+28)(F_2^-)^4 - t^2(17t+36)(F_2^-)^3$$
$$+ t(17t+19)(F_2^-)^2 - (9t+1)\,F_2^- + t + 2 \qquad (7.35)$$

The algebraic relations between $F^\pm$ and p are slightly more complicated. We assume that $p > 2/5$. The elimination of t and u_1 and u_2 between Eqs. (5.13), (5.21), and (5.9) leads to

$$0 = p^3(F^+)^6 - 10p^3(F^+)^5 + 3p^2(15p-1)(F^+)^4 - p(5p-1)(25p-1)(F^+)^3$$
$$+ 3p(15p-1)(5p-2)(F^+)^2 - 10p(5p-2)^2\,F^+ + (5p-2)^3$$
$$0 = (F^-)^6 - (10p-9)(F^-)^5 + 3(p-1)(15p-11)(F^-)^4$$
$$- (p-1)(125p^2 - 185p + 63)(F^-)^3 + 3(p-1)^2\,(15p-11)(5p-2)(F^-)^2$$
$$- (p-1)^2\,(10p-9)(5p-2)^2\,F^- + (p-1)^3\,(5p-2)^3 \qquad (7.36)$$

The above algebraic relations (7.35), (7.36) have respective degrees $\deg_t(2,5)=10$ in the $F_k^{\pm}$ and $\deg_p(2,5)=6$ in $F^{\pm}$, in agreement with Eq. (5.22). These relations also allow an investigation of the critical behavior of quantities as $p \to 2/5$. It turns out that all the critical amplitudes defined in Section 6 can be expressed as rational functions of one single irrational number

$$\zeta = 10^{1/3} \tag{7.37}$$

namely

$$(F_1^+)_c = \frac{125}{108}(2+2\zeta-\zeta^2) = 1.929723$$

$$(F_2^+)_c = \frac{3125}{1944}(2-10\zeta+5\zeta^2) = 5.889270$$

$$(F_1^-)_c = \frac{25}{18}(\zeta-1) = 1.603382, \qquad (F_2^-)_c = \frac{625}{108}(3-\zeta) = 4.893318$$

$$\phi_1^+ = \frac{625}{108} = 5.787037, \qquad \Phi_2^+ = \frac{78125}{11664}(4-2\zeta+\zeta^2) = 29.020380$$

$$\Phi_1^- = \frac{125}{36} = 3.472222, \qquad \Phi_2^- = \frac{3125}{648}(2+\zeta) = 20.034890$$

$$\Phi^+ = \frac{5}{18}(10-2\zeta+\zeta^2) = 2.870200, \qquad \Phi^- = \frac{1}{3}(5+\zeta) = 2.384812$$

$$C^+ = \frac{1}{6\sqrt{3}}(10-2\zeta+\zeta^2) = 0.994266, \qquad C^- = \frac{1}{5\sqrt{3}}(5+\zeta) = 0.826123 \tag{7.38}$$

To close up the study of this example, let us consider the duality approach of Section 5.3. We have $I_0^+ = \{1, 3\}$ and $I_0^- = \{1, 2, 4\}$. We again focus our attention on quantities with the superscript $+$. Equations (5.40), (5.41) yield four equations for the four unknown functions $\bar{F}_1^+,..., \bar{F}_4^+$:

$$\begin{aligned}
\bar{F}_1^+ &= 1 + t(\bar{F}_1^+ \bar{F}_4^+ + \bar{F}_2^+ \bar{F}_3^+) \\
\bar{F}_2^+ &= (\bar{F}_1^+)^2 + t\bar{F}_3^+ \bar{F}_4^+ \\
\bar{F}_3^+ &= \bar{F}_1^+ \bar{F}_2^+ \\
\bar{F}_4^+ &= \bar{F}_1^+ \bar{F}_3^+
\end{aligned} \tag{7.39}$$

This non-linear system can be solved in closed form. The trick is to define a variable $x = \bar{F}_2^+/(\bar{F}_1^+)^2$. The functions $\bar{F}_k^+$ and the variable t can then be expressed as rational functions of x:

$$\begin{aligned} t &= \frac{x^3(x-1)}{(x^2+x-1)^5} \\ \bar{F}_1^+ &= \frac{x^2+x-1}{x} \\ \bar{F}_2^+ &= \frac{(x^2+x-1)^2}{x} \\ \bar{F}_3^+ &= \frac{(x^2+x-1)^3}{x^2} \\ \bar{F}_4^+ &= \frac{(x^2+x-1)^4}{x^3} \end{aligned} \tag{7.40}$$

In this case, the Riemann surface of t and of the $\bar{F}_1^+,\dots,\bar{F}_4^+$ has genus 0. It is a branched covering of the t-sphere of degree $\deg_t(2, 5) = 10$, with x being the uniformizing variable. The physical region $0 \leqslant t \leqslant t_c = 108/3125$ corresponds to $1 \leqslant x \leqslant x_c = (2+\zeta^2)/6 = 1.106931$, with $\mathrm{d}t/\mathrm{d}x = 0$ at $x = x_c$. We close up this section by observing that in general the genus of the Riemann surface where all the functions $\bar{F}_k^+$ and t are uniform grows very rapidly with M and $\widetilde{M}$.

7.6. Directed Scaling Limit: $v \to 1$, i.e., $p_c \to 0$, and $p \to 0$

We now investigate the situation where p and p_c are simultaneously small. We refer to this case as the "directed scaling limit," since the random walks are almost perfectly directed toward the rightmost limit $(x = t)$ of phase space [see Fig. 1]. This nonconventional scaling limit turns out to be characterized by nontrivial scaling laws, in the three regimes described in Section 2.

We introduce the scaling variable

$$\xi = \frac{p}{p_c} \tag{7.41}$$

We consider first the convergent regime $(\xi > 1)$. The explicit results derived in Section 7.2 for the family of rational slopes with $M = 1$ exhibit scaling behavior as $p_c \ll 1$, i.e., $N \gg 1$. Indeed the small root u_1 of Eq. (5.13) scales as $u_1 \approx \xi_1 p_c$, where $0 < \xi_1 < 1$ is related to $\xi > 1$ by

$$\xi_1 e^{-\xi_1} = \xi e^{-\xi} \tag{7.42}$$

Equations (7.10), (7.11) then yield

$$F_1^\pm \approx e^{\xi_1} \tag{7.43}$$

$$F^\pm \approx (\xi - \xi_1)\, p_c \tag{7.44}$$

$$\Pi \approx \xi_1(\xi - \xi_1)\, p_c^2 \tag{7.45}$$

More generally, the algebraic formalism of Section 5.2 can be worked out for an arbitrary rational slope $p_c = M/N$ in the directed scaling limit, i.e., M finite and $N \gg 1$. The M small roots u_α of Eq. (5.13), with $\alpha = 1,..., M$, now scale as $u_\alpha \approx \xi_\alpha p_c$, where ξ_α is the solution such that $|\xi_\alpha| < 1$ of

$$\xi_\alpha e^{-\xi_\alpha} = \omega^{\alpha - 1} \xi e^{-\xi} \tag{7.46}$$

with the definition (5.15). As a consequence, solving the linear system (5.21) amounts to inverting the discrete Fourier-transform matrix $S_{k,\,l} = \omega^{kl}$ with $k, l = 1,..., M$. We have $(S^{-1})_{k,\,l} = \omega^{-kl}/M$, so that finally

$$F_k^\pm \approx p_c^{1-k} (\xi e^{-\xi})^{-k} \frac{1}{M} \sum_{\alpha=1}^{M} \omega^{-k(\alpha-1)} \xi_\alpha \tag{7.47}$$

The scaling form (7.44) for the survival probabilities $F^\pm$ thus holds true for any (rational) slope in the directed scaling limit. The probability Π still scales as p_c^2, but with a nontrivial coefficient:

$$\Pi \approx (\xi - \xi_1) \left(\frac{1}{M} \sum_{\alpha=1}^{M} \xi_\alpha \right) p_c^2 \tag{7.48}$$

The critical behavior of the survival probabilities as $p \to p_c$, i.e., $\xi \to 1$, is characterized by the amplitudes

$$\Phi^\pm \approx 2 \tag{7.49}$$

We thus obtain the scaling behavior of the amplitudes $C^\pm$ of Eq. (2.29) in the marginal regime ($p = p_c \ll 1$ or $\xi = 1$):

$$C^\pm \approx (2p_c)^{1/2} \tag{7.50}$$

in agreement with Eq. (6.16).

Some of the above results can be alternatively derived by means of the integers $A_k^\pm$. The bounds (4.10) imply that these numbers exhibit a simple scaling as $p_c \ll 1$, namely

$$A_k^\pm \approx \frac{(k/p_c)^{k-1}}{k!} \tag{7.51}$$

The behavior of the $A_k^\pm$ for large k agrees with Eq. (4.12), with a trivial modulation $B_k^\pm \approx 1$.

Equation (3.15) then yields

$$F^\pm \approx (\xi - \phi(\xi \mathrm{e}^{-\xi}))\, p_c \tag{7.52}$$

with

$$\phi(z) = \sum_{k \geqslant 1} \frac{k^{k-1} z^k}{k!} \tag{7.53}$$

This functional relation between ϕ and z is equivalent to

$$z = \phi \mathrm{e}^{-\phi} \tag{7.54}$$

with the condition that $\phi \to 0$ for $z \to 0$. The identity (7.54), which is given in refs. 9 and 10, can be checked by means of the contour integrals

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z^{k+1}}\, \phi(z) = \oint \frac{\mathrm{d}\phi}{2\pi \mathrm{i} \phi^k}\, (1-\phi)\, \mathrm{e}^{k\phi} = \frac{k^{k-1}}{k!} \tag{7.55}$$

In Eq. (7.52) we have $z = \xi \mathrm{e}^{-\xi}$ with $\xi > 1$, so that Eq. (7.42) leads to the identification $\phi = \xi_1$. The result (7.44) is thus recovered, without recourse to the algebraic trick. Its validity for any slope in the directed scaling limit, either rational or not, is thus established.

The result (7.44) can be recast as the following scaling relation for the survival probabilities

$$\frac{F^\pm}{p} \approx 1 - \exp\left(-\frac{F^\pm}{p_c}\right) \tag{7.56}$$

which is valid throughout the convergent regime of the directed scaling limit, i.e., when p, $p_c < p$, and $F^\pm$ are simultaneously small. Figure 16 illustrates this relation. The ratio F/p is plotted against p_c/p. The data for $p = 0.3$, 0.2, and 0.1 (full curves, from top to bottom), already presented in

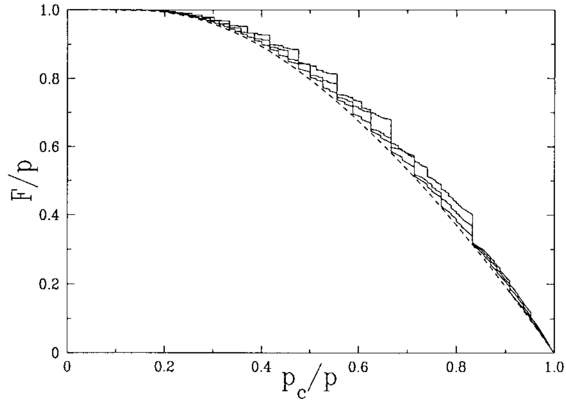

Fig. 16. Plot of F/p against p_c/p for $p=0.1$, 0.2, and 0.3, illustrating the scaling law (7.56) pertaining to the directed scaling limit, shown as a dashed curve.

Fig. 10, are found to smoothly converge toward the scaling law (7.56), shown as a dashed curve.

To close up this section, we consider briefly the large-deviation regime ($p<p_c$ or $\xi<1$). When p and p_c are simultaneously small with $p<p_c$, the survival probabilities $F^{\pm}(n, v)$ can be estimated by inserting the expression (7.51) for the integers $A_k^{\pm}$ into Eq. (6.12). We obtain after some algebra an asymptotic expression of the form (2.21), where the entropy function scales as $S(v) \approx p_c(\xi - 1 - \ln \xi)$, as it should, and where the periodic amplitudes $a^{\pm}(x)$ and $b^{\pm}(x)$ read

$$a^{\pm}(x) \approx \frac{\xi^{1-\mathrm{Frac}^{\mp}(x)}}{(2\pi p_c)^{1/2}\,(1-\xi)}, \qquad b^{\pm}(x) \approx \frac{\xi^{1-\mathrm{Frac}^{\mp}(x)}}{(2\pi p_c)^{1/2}\,(1-\xi \mathrm{e}^{1-\xi})} \tag{7.57}$$

8. DISCUSSION

We have performed a detailed analysis of the statistics of persistent events in the case of the one-dimensional lattice random walk in the presence of an obstacle moving ballistically with velocity v. Both space and time are discrete, so that the underlying lattice structure yields a highly non-trivial dependence on the velocity v of the obstacle, with discontinuities at rational values of v, for most of the quantities investigated in this work. This is especially the case for the limit survival probabilities $F^{\pm}$ in the convergent regime, and for the amplitudes $C^{\pm}$ characteristic of the marginal regime, respectively shown in Figs. 10 and 14–15. We have obtained a deep insight into the problem by means of the algebraic methods exposed in Section 5.

We want to emphasize that the statistics of persistent events can be investigated in a broader perspective, even within the realm of one-dimensional random walks, constrained to remain on the left of a moving obstacle. Consider a general random walk consisting of independent identically distributed steps ε_n, and a moving obstacle, whose position is given by an arbitrary function $X(t)$. The following situations illustrate the richness of possible behavior of this seemingly simple system [see refs. 1, 4, 11, and 12 for related questions].

- The steps ε_n have a distribution whose first two moments are finite, and the obstacle moves ballistically: $X(t) = vt$. The Sparre Andersen formalism of Section 2 applies in this situation. The survival probability is described by the scaling laws (2.21), (2.29), and (2.32), associated with the three regimes described in the introduction. The entropy function S, the constant C of the marginal regime, and the limit survival probability F in the convergent regime, are non-universal, in the sense that they depend on details of the distribution of the steps. The standard large-deviation formalism allows to determine S by means of its Legendre transform. The constant C and the limit survival probability F are given by the formal expressions (2.28) and (2.33). No closed-form expressions for these quantities are known in general. Let us mention, however, that the second expression of Eq. (2.35) is quite general, with $\operatorname{Var}\varepsilon = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$ replacing $1 - v^2$.

- The steps ε_n have a symmetric distribution whose second moment is finite, and the obstacle moves according to $X(t) = gt^{1/2}$, in units where $\langle \varepsilon \rangle^2 = 1$. In this case the survival probability of the walker in the presence of the obstacle exhibits a scaling behavior of the form

$$F(n, g) \sim n^{-\tilde{\theta}(g)} \qquad (n \gg 1) \tag{8.1}$$

where the exponent $\tilde{\theta}(g)$ is a nontrivial universal, continuously decreasing function of g, given in terms of a zero of the parabolic cylinder function.[11, 12] The marginal regime is recovered as $\tilde{\theta}(0) = 1/2$. The result (8.1) crosses over to the large-deviation regime as $g \to -\infty$, where the exponent diverges as $\tilde{\theta}(g) \approx g^2/8$, while it crosses over to the convergent regime as $g \to +\infty$, where the exponent vanishes as $\tilde{\theta}(g) \approx g e^{-g^2/2}/(8\pi)^{1/2}$. The exponent $\tilde{\theta}(g)$ is related to the persistence exponent $\theta(g)$ defined in the introduction, for the sake of consistency with refs. 1, 4, and 5, by $\tilde{\theta}(-g) = \theta(g)$.

- The steps ε_n have a symmetric broad (Lévy) distribution, with long tails falling off as $\rho(\varepsilon) \sim |\varepsilon|^{-\mu-1}$, with $0 < \mu < 2$, and the position of the obstacle obeys an asymptotic long-time behavior of the form $X(t) \approx gt^{1/\mu}$. Then the survival probability is expected to exhibit a scaling behavior of the form (8.1), again with a continuously varying exponent $\widetilde{\theta}(g)$, that is universal if the ε_n are measured in appropriate units, but whose expression is not known in general.

A simple example in this category corresponds to steps having a Cauchy distribution: $\rho(\varepsilon)=1/(\pi(1+\varepsilon^2))$, with an obstacle moving ballistically, according to $X(t)=vt$. The Sparre Andersen formalism again applies to this situation. The stability of the Cauchy law under convolutions implies that the one-time distribution function reads

$$P(n, v)=P(v)=\int_{-\infty}^{v}\frac{\mathrm{d}\varepsilon}{\pi(1+\varepsilon^2)}=\frac{1}{2}+\frac{1}{\pi}\arctan v \tag{8.2}$$

independently of n. Equation (2.1) then leads to $f(z, v)=(1-z)^{-P(v)}$, hence

$$F(n, v)=\frac{\Gamma(n+P(v))}{n!\,\Gamma(P(v))}\approx\frac{n^{-\widetilde{\theta}(v)}}{\Gamma(P(v))} \qquad (n\gg 1) \tag{8.3}$$

where $\widetilde{\theta}(v)$ is again a universal exponent, given by

$$\widetilde{\theta}(v)=1-P(v)=\frac{1}{2}-\frac{1}{\pi}\arctan v \tag{8.4}$$

The marginal regime is again recovered as $\widetilde{\theta}(0)=1/2$. The result (8.4) crosses over to the convergent regime as $v\to+\infty$, where the exponent vanishes as $\widetilde{\theta}(v)\approx 1/(\pi v)$. Contrary to the previous situation, the exponent does not cross over to the large-deviation regime as $v\to-\infty$, since it admits a finite limit $\widetilde{\theta}(-\infty)=1$. The exponent $\widetilde{\theta}(v)$ is again related to the persistence exponent $\theta(v)$ defined in the introduction by $\widetilde{\theta}(-v)=\theta(v)$, yielding in the present case $\theta(v)=P(v)$.

APPENDIX. COMBINATORIAL PROOF OF THE SPARRE ANDERSEN IDENTITY (2.1)

In this Appendix we provide an elementary and self-contained combinatorial proof of the Sparre Andersen identity (2.1).

We keep notations consistent with the body of the paper, dropping for the sake of simplicity the $\pm$ superscript and the dependence on v. Let $\{\varepsilon_n\}_{n=1,2,\dots}$ be independent identically distributed random variables, and let $x_0=0$, $x_1=\varepsilon_1,\dots$, $x_n=\varepsilon_1+\cdots+\varepsilon_n,\dots$ be their partial sums. For $n=1, 2,\dots$ we denote by $P(n)=\mathrm{Prob}\{x_n\geqslant 0\}$ the probability that the nth partial sum is non-negative, and by $F(n)=\mathrm{Prob}\{x_1\geqslant 0,\dots, x_n\geqslant 0\}$ the probability that the first n partial sums are non-negative. We set consistently $F(0)=1$.

We want to prove that the generating functions

$$p(z)=\sum_{n\geqslant 1}\frac{P(n)}{n}\,z^n, \qquad f(z)=\sum_{n\geqslant 0}F(n)\,z^n \tag{A.1}$$

are related by the Sparre Andersen identity (2.1), i.e.,

$$f(z)=\exp(p(z)) \tag{A.2}$$

For each $n\geqslant 1$, we define the event $\mathbf{A}_n$ and n events $\mathbf{B}_n^i$ for $i=1,..., n$ by

$$\begin{aligned}
\mathbf{A}_n &= \{\varepsilon_1+\cdots+\varepsilon_n\geqslant 0\}\\
\mathbf{B}_n^i &= \{\varepsilon_i\geqslant 0,\, \varepsilon_i+\varepsilon_{i+1}\geqslant 0,...,\, \varepsilon_i+\cdots+\varepsilon_n\geqslant 0,\\
&\qquad \varepsilon_i+\cdots+\varepsilon_n+\varepsilon_1\geqslant 0,...,\, \varepsilon_i+\cdots+\varepsilon_n+\varepsilon_1+\cdots+\varepsilon_{i-1}\geqslant 0\}
\end{aligned} \tag{A.3}$$

The first observation is that $\mathbf{A}_n=\bigcup_{i=1}^n \mathbf{B}_n^i$. It is clear from the definition that $\mathbf{B}_n^i\subset\mathbf{A}_n$. To prove the reverse inclusion, assume $\mathbf{A}_n$ holds, i.e., $x_n\geqslant 0$. Let $i\in\{1,..., n\}$ be such that x_{i-1} is the minimum of $\{x_0,..., x_{n-1}\}$ (in case of degeneracy, we take the smallest such i). We claim that $\mathbf{B}_n^i$ is realized. Indeed, by definition of i, $x_m-x_{i-1}\geqslant 0$ for $m\in\{i,..., n-1\}$, and (as $x_n\geqslant 0$), $x_n+(x_m-x_{i-1})\geqslant 0$ for $m\in\{0,..., i-1\}$: this covers the definition of $\mathbf{B}_n^i$.

We apply the inclusion-exclusion principle and write

$$P(n)=\mathrm{Prob}\{\mathbf{A}_n\}=\sum_{k=1}^{n}(-)^{k+1}\sum_{0<i_1<\cdots<i_k<n+1}\mathrm{Prob}\{\mathbf{B}_n^{i_1}\cap\cdots\cap\mathbf{B}_n^{i_k}\} \tag{A.4}$$

Now comes the second observation. Choose $k\geqslant 1$ and $0<i_1<\cdots<i_k<n+1$. Then

$$\begin{aligned}
&\mathbf{B}_n^{i_1}\cap\cdots\cap\mathbf{B}_n^{i_k}\\
&\quad=\{\varepsilon_{i_1}\geqslant 0,...,\, \varepsilon_{i_1}+\cdots+\varepsilon_{i_2-1}\geqslant 0\}\\
&\qquad\cap\cdots\cap\{\varepsilon_{i_j}\geqslant 0,...,\, \varepsilon_{i_j}+\cdots+\varepsilon_{i_{j+1}-1}\geqslant 0\}\\
&\qquad\cap\cdots\cap\{\varepsilon_{i_k}\geqslant 0,...,\, \varepsilon_{i_k}+\cdots+\varepsilon_n\geqslant 0,...,\, \varepsilon_{i_k}+\cdots+\varepsilon_n+\varepsilon_1+\cdots\\
&\qquad\quad+\varepsilon_{i_1-1}\geqslant 0\}
\end{aligned} \tag{A.5}$$

Indeed all the inequalities on the right-hand-side are among the defining inequalities for the left-hand-side. Moreover, it is obvious that the missing inequalities on the right-hand-side are sums of the written ones. We have thus succeeded in decomposing $\mathbf{B}_n^{i_1} \cap \cdots \cap \mathbf{B}_n^{i_k}$ as a product of k events. These events are independent because they involve distinct steps of the random walk. Another nice feature is that the event $\{\varepsilon_{i_j} \geqslant 0,..., \varepsilon_{i_j} + \cdots + \varepsilon_{i_{j+1}-1} \geqslant 0\}$ has probability $F(i_{j+1} - i_j)$. Indeed, the steps are identically distributed, so that exchanging the steps does not change probabilities, and $\{\varepsilon_{i_j} \geqslant 0,..., \varepsilon_{i_j} + \cdots + \varepsilon_{i_{j+1}-1} \geqslant 0\}$ has the same probability as $\{\varepsilon_1 \geqslant 0,..., \varepsilon_1 + \cdots + \varepsilon_{i_{j+1}-i_j} \geqslant 0\}$. We can thus rewrite Eq. (A.4) for $P(n)$ as

$$P(n) = \sum_{k=1}^{n} (-)^{k+1} \sum_{0 < i_j < \cdots < i_k < n+1} F(i_2 - i_1) \cdots F(i_k - i_{k-1}) \, F(i_1 - i_k + n) \tag{A.6}$$

The last step of the proof is as follows. Multiply Eq. (A.6) by z^n and sum over $n = 1, 2,....$ The left-hand-side yields $zp'(z)$, where the accent denotes a differentiation. At fixed k, the k-tuple sum in the right-hand-side can be performed by taking $j_1 = i_2 - i_1 \geqslant 1$, $j_2 = i_3 - i_2 \geqslant 1,..., j_{k-1} = i_k - i_{k-1} \geqslant 1$, $j_k = i_1 - i_k + n \geqslant 1$ as independent summation indices. The power of z can be recast as $z^{j_1 + \cdots + j_k}$. Each of the sums over $j_1,..., j_{k-1}$ brings a factor $f(z) - 1$, while the last sum over j_k yields $zf'(z)$. We are thus left with the equation

$$zp'(z) = \sum_{k \geqslant 1} (1 - f(z))^{k-1} \, zf'(z) \tag{A.7}$$

i.e.,

$$p'(z) = \frac{f'(z)}{f(z)} \tag{A.8}$$

which yields the identity (A.2) by integration, since $p(0) = 0$, $f(0) = 1$.

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